Extended Relativity in C-spaces: A progress report

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(Based on previous works by C.Castro and M.Pavšič)

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- C-space Relativity naturally incorporates the ideas of an invariant length (Planck scale), maximal acceleration, non-commuting coordinates, supersymmetry, holography, higher derivative gravity with torsion and variable dimensions/signatures. It permits to study the dynamics of all (closed) p-branes, for all values of p, on a unified footing!

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3. It admits superluminal propagation (tachyons) without violations of causality

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6. Objects move dilationally because of inertia
7. Higher derivative Gravity *with Torsion* in ordinary spacetime emerges naturaly from the Geometry of *curved* C-space

From Minkowski Spacetime to Clifford spaces

- Firstly, we create an extended relativity theory in C-spaces, but a natural generalization of the notion of the spacetime interval in Minkwoski space to a manifold we call C-space (C-spaces)requires extended objects.

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What is a polyvector?

The Clifford valued polyvector is a sum:

$$X = X^M E_M = \sigma \underline{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \ldots + x^{\mu_1 \mu_2 \ldots . \mu_D} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \ldots . \wedge \gamma_{\mu_D}$$

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- Interpretation: a point in a manifold, called Clifford space, C-space has coordinates X^M.
- The series of terms at a *finite* grade depending on the dimension D. A Clifford algebra $Cl(r, q)$ with $r + q = D$ has 2^D basis elements.

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$$\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}$$

- Einstein introduced the speed of light as a universal *absolute* invariant in order **to unite** space with time (to match units) in the Minkwoski space interval:

$$ds^2=c^2dt^2+dx_idx^i$$

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The C-space interval

The C-space interval generalizes Minkovskian spacetime:

$$dX^2 = d\sigma^2 + dx_\mu dx^\mu + dx_{\mu\nu} dx^{\mu\nu} + \ldots$$

From Minkowski Spacetime to Clifford spaces(III)

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1. Take the differential dX of X. Compute the scalar $< dX^\dagger dX >_0 \equiv dX^\dagger * dX \equiv |dX|^2$ and obtain the C-space extension of the particles proper time in Minkwoski space
2. The symbol $X^\dagger$ denotes the *reversion* operation and involves reversing the order of all the basis γ^μ elements in the expansion of X.It is the analog of the transpose (Hermitian) conjugation
3. The C-space metric associated with a polyparticle motion is then :

$$|dX|^2 = G_{MN}\, dX^M dX^N$$

where $G_{MN} = E_M^\dagger * E_N$ is the C-space metric.

$$|dX|^2 = d\sigma^2 + L^{-2} dx_\mu dx^\mu + L^{-4} dx_{\mu\nu} dx^{\mu\nu} + \ldots + L^{-2D} dx_{\mu_1 \ldots \mu_D}\, dx^{\mu_1 \ldots \mu_D} \tag{2}$$

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- *Neccesary* introduction: Planck scale L. It is **length** parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops,..., p-loops.
- This procedure can be carried to all closed p-branes (p-loops) where the values of p are $p = 0, 1, 2, 3, ...$ The $p = 0$ value represents the center of mass and the coordinates $x^{\mu\nu}, x^{\mu\nu\rho}...$

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- Closed membrane (2-loop) in D-dim flat spacetime is represented by the antisymmetric variables $x^{\mu\nu\rho}$:

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- Closed membrane (2-loop) in D-dim flat spacetime is represented by the antisymmetric variables $x^{\mu\nu\rho}$: *volumes* enclosed by the projections of the 2-loop along the hyperplanes of the flat target spacetime background.
- Note that D- Planck scale is $L_D = (G_N)^{1/(D-2)}$. In natural units $\hbar = c = 1$, taking the limit $D = \infty$, transform our finite L_D into $L_\infty = G^0 = 1$ (*assuming* a **finite value** of G). Conclusion: in $D = \infty$ the Planck scale has the natural value of unity. (To avoid any serious algebraic divergence problems we shall focus solely on a *finite* value of D.)

Lorentz-like polyrotations in C-spaces

C-space polyrotation

The analog of Lorentz transformations in C-spaces which transform a polyvector X into another polyvector X' is given by

$$X' = RXR^{-1}$$

Theta boosts

$$R = e^{\theta^A E_A} = exp\ [(\theta I + \theta^{\mu}\gamma_{\mu} + \theta^{\mu_1\mu_2}\gamma_{\mu_1}\wedge\gamma_{\mu_2}.....)$$
$$R^{-1} = e^{-\theta^A E_A} = exp\ [-(\theta I + \theta^{\nu}\gamma_{\nu} + \theta^{\nu_1\nu_2}\gamma_{\nu_1}\wedge\gamma_{\nu_2}.....)$$

where the theta parameters are the components of the Clifford-valued parameter $\Theta = \theta^M E_M$: $\theta; \theta^{\mu}; \theta^{\mu\nu}; ....$ and they are the C-space version of the Lorentz rotations/boosts parameters.

Lorentz-like polyrotations in C-spaces(II)

The analog of an orthogonal matrix in Clifford spaces is $R^\dagger = R^{-1}$ such that

$$< X'^\dagger X' >_s = < (R^{-1})^\dagger X^\dagger R^\dagger R X R^{-1} >_s = \\ < R X^\dagger X R^{-1} >_s = < X^\dagger X >_s = \textit{invariant}$$

$R^\dagger = R^{-1}$, will *restrict* the type of terms allowed inside the exponential defining the rotor R because the *reversal* of a p-vector obeys the identity

$$(\gamma_{\mu_1} \wedge \gamma_{\mu_2} \ldots\ldots \wedge \gamma_{\mu_p})^\dagger = \\ \gamma_{\mu_p} \wedge \gamma_{\mu_{p-1}} \ldots\ldots \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_1} = \\ (-1)^{p(p-1)/2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \ldots\ldots \wedge \gamma_{\mu_p}$$

Polyrotations and traces in C-spaces

Only those terms that change sign (under the reversal operation) are permitted in the exponential defining $R = exp[\theta^A E_A]$.

The polyvector norm

The norm of a polyvectors is defined with aid of the trace operation : $||X||^2 = Trace\ X^2$

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The norms of polyvectors transform accordingly to C-space polyrotations:
Trace $X'^2 = Trace\ [RX^2R^{-1}] = Trace\ [RR^{-1}X^2] = Trace\ X^2$
Norms (traces) are *invariant* and $RR^{-1} = 1$

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2. From Minkowski spacetime, can in general be localized not only along space-like, but also along time-like directions.E.g.: "instantonic" p-loops (space-like or time-like), long and finite (solitonic) tube-like objetcs.
3. In Minkowski spacetime M_4 –which is a subspace of C-space– we observe the intersections of Clifford lines with M_4 lines. All conservation laws hold in C-space where we have infinitely long world "lines" or Clifford,and some intersections appear as localized extended objects, p-loops, ...

The Polyparticle Dynamics in C-space: action principle

Extended object's action principle

$$I = \kappa \int d\tau \, (\dot{X}^{\dagger} * \dot{X})^{1/2} = \kappa \int d\tau \, (\dot{X}^{A} \dot{X}_{A})^{1/2}$$

where κ is a constant, "mass"-like term in C-space, and τ is an arbitrary parameter.

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The C-space velocities $\dot{X}^A = dX^A/d\tau = (\dot{\sigma}, \dot{x}^\mu, \dot{x}^{\mu\nu}, \ldots)$ are also called "holographic" velocities.

Extended object's equations of motion

$$\frac{d}{d\tau}\left(\frac{\dot{X}^A}{\sqrt{\dot{X}^B \dot{X}_B}}\right) = 0$$

Important remarks:

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- Faster than light motion is possible in C-space!

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When the denominator is zero the momentum becomes infinite. It happens when it reaches the *maximum speed* that an object accelerating in C-space can reach.

Motion in C-space(II)

Line element and polymomentum

$$dX^A dX_A = d\sigma^2 + \left(\frac{dx^0}{L}\right)^2 - \left(\frac{dx^1}{L}\right)^2 - \left(\frac{dx^{01}}{L^2}\right)^2 .... + \left(\frac{dx^{12}}{L^2}\right)^2 - \left(\frac{dx^{123}}{L^3}\right)^2 - \left(\frac{dx^{0123}}{L^4}\right)^2 + ... = 0$$

- Vanishing of $\dot{X}^B \dot{X}_B$ is equivalent to vanishing of the above C-space line element and by "..." we mean the terms with the remaining components such as x^2, x^{01}, x^{23},..., x^{012}, etc.

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- Vanishing of $\dot{X}^B \dot{X}_B$ is equivalent to vanishing of the above C-space line element and by "..." we mean the terms with the remaining components such as x^2, x^{01}, x^{23},..., x^{012}, etc.
- The C-space metric is $G_{MN} = E_M^\dagger * E_N$ and if the dimension of spacetime is 4, then x^{0123} is the highest grade coordinate.

Motion in C-space(III)

Polyvelocity

$$V^2 = -\left(L\frac{d\sigma}{dt}\right)^2 + \left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^{01}}{L^2}\right)^2 ..... - \left(\frac{1}{L}\frac{dx^{12}}{dt}\right)^2 + \left(\frac{1}{L^2}\frac{dx^{123}}{dt}\right)^2 + \left(\frac{1}{L^3}\frac{dx^{0123}}{dt}\right)^2 - ...$$

We find that the maximum speed is the maximum speed is given by $V^2 = c^2$ The maximum speed squared V^2 contains not only the components of the 1-vector velocity dx^1/dt, but also the multivector components such as dx^{12}/dt, dx^{123}/dt, ... The following special cases when only certain components of the velocity in C-space are different from zero, are of particular interest.

Motion in C-space(IV)

Maximum 1-vector speed

$$\frac{dx^1}{dt} = c = 3.0 \times 10^8 m/s$$

Maximum 3-vector speed

$$\frac{dx^{123}}{dt} = L^2 c = 7.7 \times 10^{-62} m^3/s$$

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Maximum 4-vector speed

$$\frac{dx^{0123}}{dt} = L^3 c = 1.2 \times 10^{-96} m^4/s$$

Motion in C-space(V)

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- In C-space the dynamics refers to a larger space. Minkowski space is just a subspace of C-space. So...

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- Tachyon dynamics and causality breakdown should be revised from a C-space framework!

Clifford algebra based geometric calculus in curved space(time)

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2. Let the vector fields γ_μ, $\mu = 1, 2, ..., n$ be a coordinate basis in V_n satisfying the Clifford algebra relation

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu}$$

where $g_{\mu\nu}$ is the metric of V_n. In curved space γ_μ and $g_{\mu\nu}$ cannot be constant but necessarily depend on position x^μ.An arbitrary vector is a linear superposition

$$a = a^\mu \gamma_\mu$$

where the components a^μ are *scalars* from the geometric point of view, whilst γ_μ are *vectors*.

Clifford algebra based geometric calculus in curved space(time)(II)

Vector derivative

$$\partial \equiv \gamma^{\mu}\partial_{\mu}$$

where ∂_{μ} is an operator whose action depends on the quantity it acts on

Applying the vector derivative ∂ on a *scalar* field ϕ we have

$$\partial\phi = \gamma^{\mu}\partial_{\mu}\phi$$

where $\partial_{\mu}\phi \equiv (\partial/\partial x^{\mu})\phi$ coincides with the partial derivative of ϕ. But if we apply it on a *vector* field a we have

$$\partial a = \gamma^{\mu}\partial_{\mu}(a^{\nu}\gamma_{\nu}) = \gamma^{\mu}(\partial_{\mu}a^{\nu}\gamma_{\nu} + a^{\nu}\partial_{\mu}\gamma_{\nu})$$

In general γ_{ν} is not constant; it satisfies the relation

$$\partial_{\mu}\gamma_{\nu} = \Gamma^{\alpha}_{\mu\nu}\gamma_{\alpha} \quad \partial_{\mu}\gamma^{\nu} = -\Gamma^{\nu}_{\mu\alpha}\gamma^{\alpha}$$

Covariant derivatives with Clifford calculus(I)

$$\partial a = \gamma^\mu \gamma_\nu (\partial_\mu a^\nu + \Gamma^\nu_{\mu\alpha} a^\alpha) \equiv \gamma^\mu \gamma_\nu D_\mu a^\nu = \gamma^\mu \gamma^\nu D_\mu a_\nu$$

where $_\mu$ is the covariant derivative of tensor analysis. Decomposing the Clifford product $\gamma^\mu \gamma^\nu$ into its symmetric and antisymmetric part

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$$\gamma^\mu \gamma^\nu = \gamma^\mu \cdot \gamma^\nu + \gamma^\mu \wedge \gamma^\nu$$

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$$\partial a = \gamma^\mu \gamma_\nu (\partial_\mu a^\nu + \Gamma^\nu_{\mu\alpha} a^\alpha) \equiv \gamma^\mu \gamma_\nu D_\mu a^\nu = \gamma^\mu \gamma^\nu D_\mu a_\nu$$

where $_\mu$ is the covariant derivative of tensor analysis. Decomposing the Clifford product $\gamma^\mu \gamma^\nu$ into its symmetric and antisymmetric part

$$\gamma^\mu \gamma^\nu = \gamma^\mu \cdot \gamma^\nu + \gamma^\mu \wedge \gamma^\nu$$

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$$\begin{aligned}
\partial\partial a &= \gamma^\mu \partial_\mu (\gamma^\nu \partial_\nu)(a^\alpha \gamma_\alpha) = \gamma^\mu \gamma^\nu \gamma_\alpha D_\mu D_\nu a^\alpha \\
&= \gamma_\alpha D_\mu D^\mu a^\alpha + \frac{1}{2}(\gamma^\mu \wedge \gamma^\nu)\gamma_\alpha [D_\mu, D_\nu] a^\alpha \\
&= \gamma_\alpha D_\mu D^\mu a^\alpha + \gamma^\mu (R_{\mu\rho} a^\rho + K_{\mu\alpha}{}^\rho D_\rho a^\alpha) \\
&+ \frac{1}{2}(\gamma^\mu \wedge \gamma^\nu \wedge \gamma_\alpha)(R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho D_\rho a^\alpha)
\end{aligned}$$

Curvature, torsion and final results

From the equation:

$$[D_\mu, D_\nu]a^\alpha = R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho D_\rho a^\alpha$$

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The result for **arbitrary covariant derivatives** acting onto an r-vector $A = a^{\alpha_1 \ldots \alpha_r}\gamma_{\alpha_1}\gamma_{\alpha_2}\ldots\gamma_{\alpha_r}$ is:

$$\begin{aligned}\partial\partial\ldots\partial A &= (\gamma^{\mu_1}\partial_{\mu_1})(\gamma^{\mu_2}\partial_{\mu_2})\ldots(\gamma^{\mu_k}\partial_{\mu_k})(a^{\alpha_1\ldots\alpha_r}\gamma_{\alpha_1}\gamma_{\alpha_2}\ldots\gamma_{\alpha_r}) \\ &= \gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_k}\gamma_{\alpha_1}\gamma_{\alpha_2}\ldots\gamma_{\alpha_r}D_{\mu_1}D_{\mu_2}\ldots D_{\mu_k}a^{\alpha_1\ldots\alpha_r}\end{aligned}$$

Clifford algebra based geometric calculus and resolution of the ordering ambiguity for the product of momentum operators

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If we rewrite H as $H = \frac{\Lambda}{2} p^2$ where $p = \gamma^\mu p_\mu$ is the momentum vector,we find that there is no ambiguity in writing the square p^2!When acting with H on a scalar wave function ϕ we obtain the unambiguous expression

$$H\phi = \frac{\Lambda}{2} p^2 \phi = \frac{\Lambda}{2} (-i)^2 (\gamma^\mu \partial_\mu)(\gamma^\nu \partial_\nu)\phi = -\frac{\Lambda}{2} D_\mu D^\mu \phi$$

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in which there is no curvature term R. We expect that a term with R will arise upon acting with H on a *spinor* field ψ.

Curvature of C-space and spacetime curvature

Let be a polyvector

$$A = A^A E_A = s\gamma + a^\alpha \gamma_\alpha + a^{\alpha\beta}\gamma_\alpha \wedge \gamma_\beta + ...$$

The polyderivate

$$\frac{DA^A}{DX^B} = \frac{\partial A^A}{\partial X^B} + \tilde{\Gamma}^A_{BC} A^C$$

where we defined

$$\frac{DA^A}{Dx^{\mu\nu}} = [D_\mu, D_\nu] A^A$$

Curvature of C-space and spacetime curvature(II)

$$\frac{Ds}{Dx^{\mu\nu}} = [D_\mu, D_\nu]s = K_{\mu\nu}{}^{\rho}\partial_\rho s$$

$$\frac{Da^\alpha}{Dx^{\mu\nu}} = [D_\mu, D_\nu]a^\alpha = R_{\mu\nu\rho}{}^{\alpha} a^\rho + K_{\mu\nu}{}^{\rho} D_\rho a^\alpha$$

$$\frac{Ds}{Dx^{\mu\nu}} = \frac{\partial s}{\partial x^{\mu\nu}}$$

$$\frac{Da^\alpha}{Dx^{\mu\nu}} = \frac{\partial a^\alpha}{\partial x^{\mu\nu}} + \tilde{\Gamma}^{\alpha}_{[\mu\nu]\rho} a^\rho = \frac{\partial a^\alpha}{\partial x^{\mu\nu}} + R_{\mu\nu\rho}{}^{\alpha} a^\rho$$

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where $\tilde{\Gamma}^\alpha_{[\mu\nu]\rho}$ has been identified with curvature.The dependence of coefficients s and a^α on $x^{\mu\nu}$ indicates the presence of torsion. On the contrary, when basis vectors γ_α depend on $x^{\mu\nu}$ this indicates that the corresponding vector space has non vanishing curvature.

The momentum constraint in C-space

The D=4 polyvector

$$X = \sigma + x^{\mu}\gamma_{\mu} + \gamma^{\mu\nu}\,\gamma_{\mu}\wedge\gamma_{\nu} + \xi^{\mu}\gamma_5\gamma_{\mu} + s\gamma_5$$

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The D=4 polymomentum

$$P = \mu + p^{\mu}\gamma_{\mu} + S^{\mu\nu}\gamma_{\mu}\wedge\gamma_{\nu} + \pi^{\mu}\gamma_5\gamma_{\mu} + m\gamma_5$$

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Note the absent extra terms in Minkovskian relativity

The D=4 polymomentum constraint

$$P_A P^A = \mu^2 + p_{\mu}p^{\mu} - 2S^{\mu\nu}S_{\mu\nu} + \pi_{\mu}\pi^{\mu} - m^2 = M^2$$

C-space Klein-Gordon and Dirac Wave Equations

Polymomentum correspondence principle

$$P_A \to -i\frac{\partial}{\partial X^A} = -i\left(\frac{\partial}{\partial \sigma}, \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^{\mu\nu}}, \ldots\right) \quad \Psi(x^\mu) \to \Psi(x^A)$$

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$$\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial x^\mu x_\mu} + \frac{\partial^2}{\partial x^{\mu\nu}\partial x_{\mu\nu}} + \ldots + M^2\right)\Phi = 0$$

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Note we used natural units in which $\hbar = 1, c = 1$

Clifford algebras in Phase space

In 2-dim phase space we have...

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C-phase space relations

$$e_p e_q = e_p . e_q + e_p \wedge e_q = 0 + e_p \wedge e_q = i$$

$$e_p . e_q \equiv \frac{1}{2}(e_q e_p + e_p e_q) = 0 \quad Q = q e_q + p e_q$$

$$dQdQ = (dq)^2 + (dp)^2 \Rightarrow S = m \int \sqrt{(dq)^2 + (dp)^2}$$

C-phase space action in 2n-dimensions(Nesterenko)

$$S = m \int \sqrt{(dq^\mu dq_\mu) + (\frac{L}{m})^2 (dp^\mu dp_\mu)}$$

Nesterenko action in C-spaces and Finsler geometry

Nesterenko action: alternative forms

$$S = m \int d\tau \sqrt{1 + (\frac{L}{m})^2 (dp^\mu / d\tau)(dp_\mu / d\tau)}$$

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$$S = m\int d\tau\sqrt{1+a^{-2}(d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)} = m\int d\tau\sqrt{1-\frac{g^2}{a^2}}$$

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$$d\omega = \sqrt{g_{\mu\nu}(x^\mu, dx^\mu)dx^\mu dx^\nu} \rightarrow S = m\int d\omega \leftrightarrow \text{Finsler metric}$$

Invariance under the $U(1,3)$ Group

Phase spacetime interval(Born,Low,...)

$$(d\sigma)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} =$$

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$$(d\tau)^2[1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2}] = (d\tau)^2[1 - \frac{m^2 g^2(\tau)}{m_P^2 A_{max}^2}]$$

Phase spacetime rapidities

$$\xi \equiv \sqrt{\frac{\xi_v^2}{c^2} + \frac{\xi_a^2}{b^2}} \quad \tanh\xi = \frac{ma}{m_P A_{max}}$$

where $A_{max} = c^2/L_P$ and we have a maximal force $F_{max} = m_P A_{max}$

Phase space relativistic transformations

Born's duality

$$(T, X) \rightarrow (E, P) \quad (E, P) \rightarrow (-T, -X)$$

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Composition rules for boosts

$$\xi''=\xi+\xi'\Rightarrow tanh\xi''=tanh(\xi+\xi')=\frac{tanh\xi+\tanh\xi'}{1+tanh\xi tanh\xi'}\Rightarrow$$

$$\frac{ma''}{m_PA}=\frac{\frac{ma}{m_PA}+\frac{ma'}{m_PA}}{1+\frac{m^2aa'}{m_P^2A^2}},\;\xi''_v=\xi_v+\xi'_v,\;\xi''_a=\xi_a+\xi'_a,\;\xi''=\xi+\xi'$$

Planck-Scale Areas are invariant under Acceleration Boosts

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are invariant under *infinite* acceleration boosts since
$(cosh\xi + sinh\xi)(cosh\xi - sinh\xi) = cosh^2\xi - sinh^2\xi = 1$

C-space Maxwell Electrodynamics(I)

1. C-space electrodynamics generalize Maxwell's theory:

$$F = dA, \; dF = 0$$

C-space Maxwell Electrodynamics(I)

1. C-space electrodynamics generalize Maxwell's theory:

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4. The generalized field strength in C-space is:
$$F = dA = E^M \partial_M (E^N A_N) = E^M E^N \partial_M A_N =$$
$$\frac{1}{2}\left\{E^M, E^N\right\}\partial_M A_N + \frac{1}{2}\left[E^M, E^N\right]\partial_M A_N =$$
$$\frac{1}{2}F_{(MN)}\left\{E^M, E^N\right\} + \frac{1}{2}F_{[MN]}\left[E^M, E^N\right]$$

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$$[DX]\equiv(d\sigma)(dx^0dx^1\ldots)(dx^{01}dx^{02}\ldots)\ldots.(dx^{012\ldots D})$$

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$$\int A_M dX^M = \int [DX] J_M A^M$$

C-space Maxwell Electrodynamics(IV):equations and generalizations

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$$\partial_M F^{[MN]} = J^N \quad \partial_N \partial_M F^{[MN]} = 0 = \partial_N J^N = 0$$

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$$F = DA = (dA + A \bullet A) \quad E_M \bullet E_N = E_M E_N - (-1)^{s_M s_N} E_N E_M$$

The Cl(1,3) algebra in D=4

The Clifford $Cl(1,3)$ algebra associated with the tangent space of a $4D$ spacetime $\mathcal{M}$ is defined by $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$ such that

$$[\Gamma_a, \Gamma_b] = 2\Gamma_{ab}, \;\; \Gamma_5 = -i\,\Gamma_0\,\Gamma_1\,\Gamma_2\,\Gamma_3, \;\; (\Gamma_5)^2 = 1; \;\; \{\Gamma_5, \Gamma_a\} = 0$$

$$\Gamma_{abcd} = \epsilon_{abcd}\,\Gamma_5; \quad \Gamma_{ab} = \frac{1}{2}(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a)$$

$$\Gamma_{abc} = \epsilon_{abcd}\,\Gamma_5\,\Gamma^d; \quad \Gamma_{abcd} = \epsilon_{abcd}\,\Gamma_5$$

$$\Gamma_a\,\Gamma_b = \Gamma_{ab} + \eta_{ab}; \quad \Gamma_{ab}\,\Gamma_5 = \frac{1}{2}\epsilon_{abcd}\,\Gamma^{cd}$$

$$\Gamma_{ab}\,\Gamma_c = \eta_{bc}\,\Gamma_a - \eta_{ac}\,\Gamma_b + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d;$$

$$\Gamma_c\,\Gamma_{ab} = \eta_{ac}\,\Gamma_b - \eta_{bc}\,\Gamma_a + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d$$

$$\Gamma_a\,\Gamma_b\,\Gamma_c = \eta_{ab}\,\Gamma_c + \eta_{bc}\,\Gamma_a - \eta_{ac}\Gamma_b + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d$$

$$\Gamma^{ab}\,\Gamma_{cd} = \epsilon^{ab}{}_{cd}\,\Gamma_5 - 4\delta^{[a}_{[c}\,\Gamma^{b]}{}_{d]} - 2\delta^{ab}_{cd};\; \delta^{ab}_{cd} = \frac{1}{2}(\delta^a_c\,\delta^b_d - \delta^a_d\,\delta^b_c)$$

The Cl(1,3) polyvectors

The C-space antihermitian gauge field 1-form

$$\mathbf{A} = \left(i\, a_\mu\, \mathbf{1} + i\, b_\mu\, \Gamma_5 + e^a_\mu\, \Gamma_a + i\, f^a_\mu\, \Gamma_a\, \Gamma_5 + \frac{1}{4}\omega^{ab}_\mu\, \Gamma_{ab} \right) dx^\mu$$

The C-space strength 2-form

$$F_{\mu\nu} = i\, F^1_{\mu\nu}\, \mathbf{1} + i\, F^5_{\mu\nu}\, \Gamma_5 + F^a_{\mu\nu}\, \Gamma_a + i\, F^{a5}_{\mu\nu}\, \Gamma_a\, \Gamma_5 + \frac{1}{4} F^{ab}_{\mu\nu}\, \Gamma_{ab}$$

where $F = \frac{1}{2}\, F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

2-form C-space curvature

The C-space strength 2-form components

$$F^1_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu ; F^5_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu + 2e^a_\mu f_{\nu a} - 2e^a_\nu f_{\mu a}$$

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$$F^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu \omega_{\nu c}{}^{b} + 4\left(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu\right) - \mu \longleftrightarrow \nu.$$

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We have obtained Maxwell electromagnetism *plus* additional extra terms

The Cl(1,3) scalar matter field

The C-space dimensionless anti-Hermitian scalar matter field polyvector

$$\Phi(x^\mu) = \Phi^A(x^\mu)\ \Gamma_A$$

$$\Phi \ = \ i\ \phi^{(1)}\ \mathbf{1} \ + \ \phi^a\ \Gamma_a \ + \ \phi^{ab}\ \Gamma_{ab} \ + i\ \phi^{a5}\ \Gamma_a\ \Gamma_5 \ + i\ \phi^{(5)}\ \Gamma_5$$

so that the covariant exterior differential is

$$d_A\ \Phi \ = \ (d_A\ \Phi^C)\ \Gamma_C \ = \ \left(\ \partial_\mu\ \Phi^C \ + \ \mathcal{A}_\mu^A\ \Phi^B\ f_{AB}{}^C\ \right)\ \Gamma_C\ dx^\mu$$

where

$$[\mathcal{A}_\mu,\ \Phi] \ = \ \mathcal{A}_\mu^A\ \Phi^B\ [\Gamma_A,\ \Gamma_B] \ = \ \mathcal{A}_\mu^A\ \Phi^B\ f_{AB}{}^C\ \Gamma_C$$

The C(1,3) actions

The C-space scalar piece of the action

$$I_1 = \int_{M_4} d^4x \ \epsilon^{\mu\nu\rho\sigma} < \Phi^A \ F^B_{\mu\nu} \ F^C_{\rho\sigma} \ \Gamma_A \ \Gamma_B \ \Gamma_C >_0$$

where the operation $< .... >_0$ denotes taking the *scalar* part of the Clifford geometric product of $\Gamma_A \ \Gamma_B \ \Gamma_C$

The C-space Chern-Simons type piece

$$I_2 = \int_{M_4} < \Phi^E \ d\Phi^A \wedge d\Phi^B \wedge d\Phi^C \wedge d\Phi^D \ \Gamma_{[E} \ \Gamma_A \ \Gamma_B \ \Gamma_C \ \Gamma_{D]} >_0$$

The C(1,3) actions(II)

The Higgs potential type piece

$$I_3 = -\int_{M_5} < d\Phi^A \wedge d\Phi^B \wedge d\Phi^C \wedge d\Phi^D \wedge d\Phi^E \Gamma_{[A}\Gamma_B \Gamma_C \Gamma_D \Gamma_{E]} >_0 \mathbf{V} =$$

$$-\int_{M_5} d\Phi^5 \wedge d\Phi^a \wedge d\Phi^b \wedge d\Phi^c \wedge d\Phi^d \ \epsilon_{abcd} \ V(\Phi) \ + \ \ldots$$

where

$$\mathbf{V} = V(\Phi) \ = \ \kappa \ \left(\ \Phi_A \ \Phi^A \ - \mathbf{v}^2 \ \right)^2$$

and

$$\Phi_A \ \Phi^A \ = \ \phi^{(1)} \ \phi_{(1)} + \phi^a \ \phi_a + \phi^{ab} \ \phi_{ab} + \phi^{a5} \ \phi_{a5} + \phi^{(5)} \ \phi_{(5)}$$

Final action

The total action in D=4 is then

$$I_1+I_2+I_3 = \frac{4}{5}\mathbf{v}\int_M d^4x \left(F^{ab}\wedge F^{cd}\epsilon_{abcd} + F^{(1)}\wedge F^{(5)} + F^{a}\wedge F^{a5}\right) =$$

$$\frac{4}{5}\,\mathbf{v}\int_M d^4x\,\left(F^{ab}_{\mu\nu}\,F^{cd}_{\rho\sigma}\,\epsilon_{abcd} + F^{(1)}_{\mu\nu}\,F^{(5)}_{\rho\sigma} + F^{a}_{\mu\nu}\,F^{a5}_{\rho\sigma}\right)\,\epsilon^{\mu\nu\rho\sigma}$$

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Einsteins's convention is assumed on indices

Field equations

Varying $I_1 + I_2 + I_3$ w.r.t the remaining scalars $\phi^1, \phi^a, \phi^{ab}, \phi^{a5}$, and taking into account the v.e.v which minimize the potential,

$$< \phi^{(5)} >= \mathbf{v}; < \phi^{(1)} >=< \phi^a >=< \phi^{ab} >=< \phi^{a5} >= 0$$

Field equations

$$2\, F^a{}_b \wedge F^b{}_a + F^{(1)} \wedge F^{(1)} + F^{(5)} \wedge F^{(5)} + F^a \wedge F_a + F^{a5} \wedge F_{a5} \; = \; 0$$

$$F^{(1)} \wedge F^a \; + \; F^{ab} \wedge F^c \; \eta_{bc} \; = \; 0$$

$$F^{(1)} \wedge F_{ab} \; + \; F^c \wedge F^{d5} \; \epsilon_{abcd} \; = \; 0$$

$$F^{(1)} \wedge F_{a5} \; + \; F^{bc} \wedge F^d \; \epsilon_{abcd} \; = \; 0$$

Generalized polyvector valued gauge fields

$$\mathbf{X} = \varphi\,\mathbf{1} + x_\mu\,\gamma^\mu + x_{\mu_1\mu_2}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2} + x_{\mu_1\mu_2\mu_3}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\gamma^{\mu_3} + \ldots$$

E.g.: Polyvector valued gauge field in Cl (5,C)

$\mathcal{A}_M(\mathbf{X}) = A^I_M(\mathbf{X})\,\Gamma_I$ is spanned by $16+16$ generators. The expansion of the poly-vector $\mathcal{A}^I_M$ is also of the form

$$\mathcal{A}^I_M = \Phi^I\,\mathbf{1} + A^I_\mu\,\gamma^\mu + A^I_{\mu_1\mu_2}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2} + A^I_{\mu_1\mu_2\mu_3}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\gamma^{\mu_3} + \ldots$$

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Summary and conclusions:C-space as an alternative tool for unification

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- Polyvector actions as toy models and relations the Standard Model. Although Cl(1,3) is not enough for unification with gravity, further Clifford algebras and groups could accomplish it!

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- Gravity plus SM from a C-space action without or with dimensional reduction.
- Generalized description of p-branes from Clifford spaces.
- Finsler like geometry and reciprocal relativity in C-spaces.
- Model building and predictions/explanations of phenomena from Clifford space relativity.
- Relation with quaternionic, octonionic, doubly special relativities...

References and related work

This work is based on the review:
" THE EXTENDED RELATIVITY THEORY IN CLIFFORD SPACES" by C. Castro, M. Pavšič *Progress in Physics* **1** (2005) 31

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