Multitemporal relativity and Unconventional Relativistic theories

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Abstract

We provide an introduction to multitemporal relativity and other multitemporal unconventional relativitistic theories.

1 Introduction

Multitemporal Special Relativity was introduced by N. S. Kalitzin, via the line element

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - c_1^2 dt_1^2 - c_2^2 dt_2^2 - \ldots - c_{n-3}^2 dt_{n-3}^2 \tag{1}$$

or

$$ds_2(kal) = \sum_{i=1}^{3} dx_i - \sum_{j=1}^{n-3} c_j^2 t_j^2 \tag{2}$$

Usually 3+1 SR can be recovered in the case $c_2 = c_3 = \cdots = c_{n-3} = 0$ and $c_1 = c$. We know define

$$-V_1^2 = \frac{ds^2}{dt_1^2} = \left(\frac{dx_1}{dt_1}\right)^2 + \left(\frac{dx_2}{dt_1}\right)^2 + \left(\frac{dx_3}{dt_1}\right)^2 - c_1^2 - \left(\frac{c_2 dt_2}{dt_1}\right)^2 - \left(\frac{c_{n-3} dx_{n-3}}{dt_1}\right)^2 \tag{3}$$

Also, we can rewrite this as follows

$$-\frac{v_1^2}{c_1^2} = \frac{ds^2}{c_1^2 dt_1^2} = \left(\frac{dx_1}{c_1^2 dt_1}\right)^2 + \left(\frac{dx_2}{c_1^2 dt_1}\right)^2 + \left(\frac{dx_3}{c_1^2 dt_1}\right)^2 - 1 - \left(\frac{c_2 dt_2}{c_1 dt_1}\right)^2 - \left(\frac{c_{n-3} dx_{n-3}}{c_1 dt_1}\right)^2 \tag{4}$$

and

$$\frac{v_1^2}{c_1^2} = \frac{ds^2}{c_1^2 dt_1^2} = 1 - \left(\frac{dx_1}{c_1^2 dt_1}\right)^2 + \left(\frac{dx_2}{c_1^2 dt_1}\right)^2 + \left(\frac{dx_3}{c_1^2 dt_1}\right)^2 + \sum_{j=2}^{n-3} \left(\frac{c_j}{c_1}\right)^2 \left(\frac{dt_j}{dt_1}\right)^2 \tag{5}$$

$$\frac{v_1^2}{c_1^2} = 1 - \left(\frac{v^2}{c_1^2}\right)^2 + \sum_{j=2}^{n-3} \left(\frac{c_j}{c_1}\right)^2 \left(\frac{dt_j}{dt_1}\right)^2 \tag{6}$$

and thus

$$\left(\frac{v_1^2}{c_1^2}\right) = \frac{1}{\Gamma^2} \tag{7}$$

$$v_1 = \frac{c_1}{\Gamma} \tag{8}$$

where

$$\Gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c_1}\right)^2 + \sum_{j=2}^{n-3} \left(\frac{c_j}{c_1}\right)^2 \left(\frac{dt_j}{dt_1}\right)^2}} \tag{9}$$

Using 9, we can obtain several limits:

- $c_j = 0 \forall j$. Space theory, newtonian relativity with no time coordinates.

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- $c_1 = c$, $c_j = 0 \forall j \neq 1$. Usual 3+1d special relativity.
- $c_j = c \forall j$. We get time isotropic multitemporal relativity with maximal speed ($c_1 = c$)

$$V_1 = c_1 \sqrt{1 + \sum_{k=2}^{n-3} \frac{c_k^2}{c_1^2}} = \sqrt{n-3}c \tag{10}$$

In general, for $s \neq k$-time coordinate, we have a group velocity

$$V_s = \frac{ds}{dt_s} = \sqrt{1 + \sum_{k \neq s} \left(\frac{c_k dt_k}{c_s dt_s} \right)^2} c_s \tag{11}$$

It can be shown that

$$u_k^r u_k^s = -c_s^2 \delta_{rs} \tag{12}$$

with $u_k^s = ds_k / d\tau_s = \Gamma ds_k / dt_s$. We can always define

$$c_4 t_4 = L_4, c_5 t_5 = L_5, \ldots, c_n t_n = L_n \tag{13}$$

and impose that extra time-like timensions are so tiny that can not be seen or conflict with causality. A different issue is the quantum stability of extra time-like dimensions.

$$w_\alpha^2 = w^2 = \frac{1}{\left(\frac{d\bar{l}_\alpha}{d\bar{x}} \right)^2} \tag{14}$$

$$w_\alpha = \frac{\left(\frac{d\bar{l}_\alpha}{d\bar{x}} \right)}{\sum_{\alpha=4}^{n} \left(\frac{d\bar{l}_\alpha}{d\bar{x}} \right)} \tag{15}$$

Multitemporal Lorentz transformatios are

$$x' = \frac{1}{\sqrt{1-w^2}} \left(x - w^2 l_\alpha \left(\frac{d\bar{l}_\alpha}{d\bar{x}} \right) \right) \tag{16}$$

and the inverse

$$x = \frac{1}{\sqrt{1-w^2}} \left(x' + w^2 l'_\alpha \left(\frac{d\bar{l}_\alpha}{d\bar{x}} \right) \right) \tag{17}$$

2 Reformulation and new notation

I will be using a new multitemporal notation from now. Let it be the multitemporal spacetime vector

$$\mathbb{X} = (x, y, z, c_4 t_4, c_5 t_5, \ldots, c_N t_N) = (\vec{r}, \vec{ct}) \tag{18}$$

$$\mathbb{X}^2 = s^2 = c_4^2 t_4^2 + c_5^2 t_5^2 + \cdots + c_N^2 t_N^2 - x^2 - y^2 - z^2 \tag{19}$$

and

$$ds^2 = -dx^2 - dy^2 - dz^2 + c_4^2 dt_4^2 + \cdots + c_N^2 dt_N^2 \tag{20}$$

or

$$ds^2 = c_4^2 dt_4^2 \left(1 + \sum_{\alpha=5}^{n} \left(\frac{c_\alpha dt_\alpha}{c_4 dt_4} \right)^2 \right) - c_4^2 dt_4^2 \left(\left(\frac{dx}{c_4 dt_4} \right)^2 + \left(\frac{dy}{c_4 dt_4} \right)^2 + \left(\frac{dz}{c_4 dt_4} \right)^2 \right) \tag{21}$$

Defining

$$v_i = \frac{dx_i}{dt_4} \tag{22}$$

$$ds^2 = c_4^2 dt_4^2 \left(1 + \sum_{\alpha=5}^{n} \left(\frac{c_\alpha dt_\alpha}{c_4 dt_4}\right)^2 - \frac{v^2}{c_4^2}\right) = c_4^2 dt_4^2 \frac{1}{\Gamma^2} \tag{23}$$

Let us write

$$u_i = \frac{dx_i}{ds},\ i = 1, \ldots, n \tag{24}$$

$$u_\rho = \frac{v_\rho \Gamma}{c_4},\ \rho = 1, 2, 3 \tag{25}$$

$$u_4 = i\Gamma \tag{26}$$

$$u_5 = i\frac{c_5 dt_5}{c_4 dt_4}\Gamma \tag{27}$$

$$\vdots \tag{28}$$

$$u_n = i\frac{c_n dt_n}{c_4 dt_4}\Gamma \tag{29}$$

$$u_i^2 = -1 \tag{30}$$

and the group velocity in multitemporal relativity reads

$$v_g^2 = c_4^2 + \sum_{\alpha=5}^{n} \left(\frac{c_\alpha dt_\alpha}{dt_4}\right)^2 \tag{31}$$

Now, as before, if $dt_4 = \cdots = dt_n = dt$ and $c_4 = \cdots = c_n = c$, the isotropy of time-like coordinates provide

$$v_g = V_{max} = \sqrt{c_4^2 + \sum_{\alpha=5}^{n} \left(\frac{c_\alpha dt_\alpha}{dt_4}\right)^2} = \sqrt{n-3}c \tag{32}$$

with

$$\Gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c_4}\right)^2 + \sum_{j=5}^{n} \left(\frac{c_j}{c_4}\right)^2 \left(\frac{dt_j}{dt_4}\right)^2}} \tag{33}$$

3 Multitemporal length contraction

In the multitemporal theory of relativity (MTHOR), the contraction length is generalized to the following formula

$$L = L_0\sqrt{1-\omega^2} = L_0 \sqrt{\frac{1 + \left(\frac{c_5 d\bar{t}_5}{c_4 dt_4}\right)^2 + \cdots + \left(\frac{c_n d\bar{t}_n}{c_4 dt_4}\right)^2 - \frac{v^2}{c_4^2}}{1 + \left(\frac{c_5 d\bar{t}_5}{c_4 dt_4}\right)^2 + \cdots + \left(\frac{c_n d\bar{t}_n}{c_4 dt_4}\right)^2}} \tag{34}$$

or equivalently

$$L = L_0\sqrt{1-\omega^2} = L_0 \sqrt{\frac{1 + \sum_{\alpha=5}^{n} \left(\frac{c_\alpha d\bar{t}}{c_4 dt_4}\right)^2 - \frac{v^2}{c_4^2}}{1 + \sum_{\alpha=5}^{n} \left(\frac{c_\alpha d\bar{t}}{c_4 dt_4}\right)^2}} \tag{35}$$

4 Multitemporal time dilation

In MTHOR, the time dilation can be generalized from unitemporal relativity into the next formula

$$\Delta t_4 = \Delta t' \left[1 + \frac{1}{1+\sum_{\alpha=5}^{n}\left(\frac{c_\alpha d\bar{t}_\alpha}{c_4 dt_4}\right)^2} \left(\sqrt{\frac{1+\sum_{\alpha=5}^{n}\left(\frac{c_\alpha d\bar{t}_\alpha}{c_4 dt_4}\right)^2}{1+\sum_{\alpha=5}^{n}\left(\frac{c_\alpha d\bar{t}_\alpha}{c_4 dt_4}\right)^2 - \frac{v^2}{c_4^2}}} - 1 \right) \right] \tag{36}$$

5 Multitemporal addition of velocities

In MTHOR, the relativistic addition of velocities is more complex than the one in unitemporal relativity. It reads

$$V = \frac{\left[\frac{v'_x}{c_4} + \frac{v}{c_4} \frac{1}{\left(1+\sum_{\alpha=5}^{n}\frac{c_\alpha^2 dt_\alpha^2}{c_4^2 dt_4^2}\right)} \left(1+\sum_{\alpha=5}^{n}\frac{c_\alpha^2 dt'_\alpha d\bar{t}_\alpha}{c_4^2 dt'_4 d\bar{t}_4}\right) \right]}{\sqrt{1-\omega^2} + \left(\frac{1+\sum_{\alpha=5}^{n}\frac{c_\alpha^2 dt'_\alpha d\bar{t}_\alpha}{c_4^2 dt'_4 d\bar{t}_4}}{1+\sum_{\alpha=5}^{n}\frac{c_\alpha^2 d\bar{t}_\alpha^2}{c_4^2 d\bar{t}_4^2}} \right)\left(1-\sqrt{1-\omega^2}\right) + \frac{v'_x v}{c_4^2}\frac{1}{\left(1+\sum_{\alpha=5}^{n}\frac{c_\alpha^2 d\bar{t}_\alpha^2}{c_4^2 d\bar{t}_4^2}\right)}} \tag{37}$$

Other consequences of multitemporal theories:

- Energy becomes a vector, not an scalar, quantity.
- Crystalline relativity or quasicrystalline relativity (due to discrete time vectors).
- Further deformation of dispersion relation between energy and momentum.
- Generalized Maxwell equations and Einstein Field Equations.
- Object invisibility from certain spacetime points.

6 Multispace gravity

In any Dd ($D = d + 1$) Universe (spacetime), the gravitational force, the gravitational field, the potential energy and the potential read

$$F_N = G_D \frac{Mm}{r^{D-2}} = G_{d+1}\frac{Mm}{r^{d-1}} \quad g = G_D\frac{M}{r^{D-2}} = G_{d+1}\frac{M}{r^{d-1}} \tag{38}$$

$$U_g = G_D\frac{Mm}{r^{D-3}} = G_{d+1}\frac{Mm}{r^{d-2}} \quad ; \quad V_g = G_D\frac{2\Gamma((D-1)/2)M}{\pi^{(D-3)/2}r^{D-3}} = G_{d+1}\frac{2\Gamma(d/2)M}{\pi^{(d-2)/2}(d-2)r^{d-2}} \tag{39}$$

Dilution of gravity: $G_N(4d) = G_D/V_D$. $g_{YM}^2(4d) = g_{YM,d}^2 R^{-d}$, and $M_P = \sqrt{hc/G} \sim 10^{-5}g$, with $M_W = \frac{h}{c}\sqrt{\Lambda/3} \sim 10^{-65}g$. Moreover, $Gh\Lambda/c^3 \sim 10^{-121}$. $M_U = \frac{c^2}{G}\sqrt{3/\Lambda} \sim 10^{56}g$, with $M'_W = \sqrt[3]{\frac{h^2\sqrt{\Lambda/3}}{G}} \sim 10^{-25}g$. You get $M_U/M_W \sim 10^{121}$.

Gravity can be seen as an entropic force (Verlinde). Hypothesis for $D = d + 1$ hyperdimensional Newton gravity:

- $A(\Sigma) = \dfrac{2\pi^{d/2}R^{d-1}}{\Gamma(d/2)}$.
- $N = A(\Sigma)/L_p^{d-1}$, $E = mc^2 = Nk_BT/2$, $\Delta S = 2\pi k_B\dfrac{mc\Delta x}{\hbar}$.

Then:

$$F = -T\frac{\Delta S}{\Delta x} = -G_d\frac{Mm}{R^{d-1}}$$

where

Hyperdimensional gravitational Newton constant

$$G_d = \frac{2\pi^{1-d/2}\Gamma\left(\frac{d}{2}\right)c^3L_p^{d-1}}{\hbar} = 2\pi^{1-d/2}\Gamma\left(\frac{d}{2}\right)\frac{c^3L_p^{d-1}}{\hbar}$$

$$\phi_g = -\Omega_dG_dM; \quad \phi_e = \Omega_dK_dQ = Q/\varepsilon_0(d) \quad ; \quad \Omega_d = 2\pi^{d/2}/\Gamma(d/2)$$

7 Multitemporal gravity

Gravitational theory can be made multitemporal as well. In the case of newtonian gravity, its multitemporal generalization ($N = n+1+d$, N is the number of total dimensions of spacetime in multitime)

$$F = G\cos^2\theta\frac{m_1m_2}{R^d} \tag{40}$$

where

$$\cos\theta = \vec{n}_1\cdot\vec{n}_2 \tag{41}$$

is the angle between the time vectors of the n-dimensional time manifold. Note that the effective gravitational constant

$$G_{eff} = G\cos^2\theta \geq 0 \tag{42}$$

and for $\theta = \pi/2$, $G_{eff} = 0$. So, gravity can be absent from some multitemporal submanifolds.

8 Multitemporal mechanics(I)

Usual 1T newtonian physics: $F = ma = m\dfrac{dv}{dt} = m\dfrac{d^2r}{dt^2} = -\nabla U(r)$, assuming conservative forces only. Let $W = F_idx^i$ the work form, in a ND manifold $V\subset\mathbb{R}^N$, with submanifold nd $M\subset\mathbb{R}^n\subset\mathbb{R}^N$. $y^I = y^I(x)$, $\omega = f_Idy^I$ implies $dy^I = \dfrac{\partial y^I}{\partial x^i}dx^i$, and also

$$W = F_i(x)dx^i \rightarrow F_I = f_I(y(x))\frac{\partial y^I}{\partial x^i}$$

Single time manifold approach

$$f_I = m\delta_{IJ}\frac{d\dot{y}^J}{dt} = m\delta_{IJ}\frac{d^2y^J}{dt^2}$$

$$F_i = m\delta_{IJ}\frac{d\dot{y}^I}{dt}\frac{\partial y^J}{\partial x^i} = m\delta_{IJ}\frac{d^2y^I}{dt^2}\frac{\partial y^J}{\partial x^i}$$

9 Multitemporal mechanics(II)

Going multitemporal with timelike coordinates $(t) = t^\alpha$, $\alpha = 1, \cdots, m$

Multitime tensorial Newton 2nd law

$$f_I = m\delta_{IJ}\delta^{\alpha\beta}\frac{\partial^2 y^J}{\partial t^\alpha \partial t^\beta}$$

$$f_i = m\delta_{IJ}\delta^{\alpha\beta}\frac{\partial^2 y^I}{\partial t^\alpha \partial t^\beta}\frac{\partial y^J}{\partial x^i}$$

with anti-trace $F_i = F^\alpha_{i\alpha}$ given by the tensor 1-form

$$F^\sigma_{i\alpha} = m\delta l_{IJ}\delta^{\sigma\beta}\frac{\partial^2 y^I}{\partial t^\alpha \partial t^\beta}\frac{\partial y^J}{\partial x^i}$$

(Multitime) Kinetic energy

$$T = E_k = \frac{1}{2}m\delta_{IJ}\dot{y}^I\dot{y}^J \quad T = \frac{1}{2}\delta_{IJ}\delta^{\alpha\beta}\frac{\partial y^I}{\partial t^\alpha}\frac{\partial y^J}{\partial t^\beta}$$

10 Multitemporal mechanics(III)

Single time Euler-Lagrange 1st order EOM

$$\delta S = 0 \rightarrow E(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0$$

11 Multitemporal Hamilton-Jacobi

The multitemporal Hamilton-Jacobi can be written in Kalitzian relativity as follows

$$\sum_{i=1}^{2}\frac{\partial^2 S}{\partial x_i^2} - \sum_{j=4}^{n}\frac{\partial^2 S}{\partial t_j^2} - m_0 c_4^2 = 0 \tag{43}$$

(Multitime) Euler-Lagrange EOM

$$\delta S = 0 \rightarrow E(L) = \frac{\partial L}{\partial x^i} - D_\alpha\left(\frac{\partial L}{\partial D_\alpha x^i}\right) = 0$$

(Multitime) Euler-Lagrange EOM: nth order

$$E(L) = \sum_{j=0}^{n}(-1)^j\left(\frac{\partial L}{\partial \partial_t^j x^i}\right) = 0 \quad E(L) = \sum_{J=0}^{n}(-1)^J\left(\frac{\partial^J L}{\partial D_\alpha^J x^i}\right) = 0$$

12 Multitemporal mechanics(IV)

Single time Hamilton EOM

Define 1T hamiltonian as $H = \dot{x}^i \dfrac{\partial L}{\partial \dot{x}^i} - L$, and $p_i = \partial L / \partial \dot{x}^i$, then

$$\dot{x}^i = \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}$$

Multi-time Hamilton EOM

Define nT hamiltonian as $H = D_\alpha x^i \dfrac{\partial L}{\partial D_\alpha x^i} - L$, and $p_i^\alpha = \partial L / \partial D_\alpha x^i$, then

$$\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p_i^\alpha} \qquad \frac{\partial p_i^\beta}{\partial t^\alpha} = -\delta^\beta{}_\alpha \frac{\partial H}{\partial x^i}$$

13 Towards multitemporal multivectors and branes

Point particles have being generalized into objects we call branes (or p-branes). The electromagnetic forms coupling to p-branes are given by

$$A = A_\mu dx^\mu \tag{44}$$
$$A_2 = B_{\mu_1\mu_2} dx^{\mu_1} \wedge dx^{\mu_2} \tag{45}$$
$$A_3 = C_{\mu_1\mu_2\mu_3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \tag{46}$$
$$\vdots \tag{47}$$
$$A_p = A_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{48}$$

The metric tensor for p-branes could suggestively be related to extended metrics

$$ds^2 = d\sigma^2 \tag{49}$$
$$ds^2 = g_{\mu_1\mu_2} dx^{\mu_1} \otimes dx^{\mu_2} \tag{50}$$
$$ds^3 = h_{\mu_1\mu_2\mu_3} dx^{\mu_1} \otimes dx^{\mu_2} \otimes dx^{\mu_3} \tag{51}$$
$$ds^4 = K_{\mu_1\mu_2\mu_3\mu_4} dx^{\mu_1} \otimes dx^{\mu_2} \otimes dx^{\mu_3} \otimes dx^{\mu_4} \tag{52}$$
$$\vdots \tag{53}$$
$$dx^N = g_{\mu_1\cdots\mu_N} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_N} \tag{54}$$

In the so-called Clifford spaces, we have a nice expansion for multivector metrics

$$ds^2 = g_{AB} dX^A dX^B = d\sigma^2 + dx_\mu dx^\mu + \cdots + dx_{\mu_1\cdots\mu_D} dx^{\mu_1\cdots\mu_D} \tag{55}$$

14 Anisotropic Relativity

Finslerian relativity in anisotropic relativity (AR) has been studied by several authors. 1+1 AR Lorent-type transformations are

$$x' = \left(\frac{1-\beta}{1+\beta}\right)^{b/2} \gamma(x - \beta t) \tag{56}$$

$$t' = \left(\frac{1-\beta}{1+\beta}\right)^{b/2} \gamma(t - \beta t) \tag{57}$$

15 Maximal acceleration and beyond

Maximal acceleration can be added as hypothesis in a finsler-like geometry $ds^2 = g_{\mu\nu}(x,\dot{x})dx^\mu dx^\nu$. And a generalized gamma factor arises when $g(\ddot{x},\ddot{x}) < a_M^2$:

$$\Gamma(v,a) = \frac{1}{\sqrt{1-a^2/a_M^2}}\frac{1}{\sqrt{1-v^2/c^2}} \tag{58}$$

A natural higher order (maximal, lenght, maximal velocity, maximal acceleration, maximal jerk, maximal snap, maximal pop,...) gamma should be like this

$$\Gamma(X,V,A,\ldots,\partial^n X) = \frac{1}{\sqrt{1-l^2/L_\Lambda^2}}\frac{1}{\sqrt{1-v^2/c^2}}\frac{1}{\sqrt{1-a^2/a_M^2}}\frac{1}{\sqrt{1-j^2/j_m^2}}\cdots \tag{59}$$

or

$$\Gamma(X,\partial X,\ldots,\partial^n X) = \prod_{i=0}^{n}\frac{1}{\sqrt{1-(\partial^i X)^2/x_{im}^2}} \tag{60}$$

16 Ultrareferential Minimal Velocity Relativity

Claudio Nassif has built a modified relativity with maximal AND minimal velocity due to the quantum realm and some ultrareferential. The modified gamma factor is now

$$\Gamma(v,V) = \frac{\sqrt{1-V^2/v^2}}{\sqrt{1-v^2/c^2}} \tag{61}$$

By the same arguments of the previous section, one could generalize this stuff to include maximal and minimal $\partial_i X$, such as

$$\Gamma(x,X,v,V,a,A,\ldots,\partial_i x,\partial_j X) = \frac{\sqrt{1-L_p^2/x^2}}{\sqrt{1-X^2/L_\Lambda^2}}\frac{\sqrt{1-V_0^2/v^2}}{\sqrt{1-V^2/c^2}}\frac{\sqrt{1-A_0^2/a^2}}{\sqrt{1-A^2/a_M^2}}\frac{\sqrt{1-J_m^2/j^2}}{\sqrt{1-J^2/j_m^2}}\cdots \tag{62}$$

or

$$\Gamma(X,\partial X,\ldots,\partial^n X) = \prod_{i=0}^{n}\frac{\sqrt{1-X_0^{i2}/\partial_i^2 x}}{\sqrt{1-\partial_i^2 X/x_{im}^2}} \tag{63}$$

17 Split octonion special relativity

Merab Gogberashvili introduced the split octonion relativity with invariant

$$s = ct + x^n J_n + \hbar\lambda^n j_n + c\hbar\omega I, \quad n = 1,2,3,\ldots, J_n^2 = I^2 = 1, j_n^2 = -1 \tag{64}$$

and

$$s^+ = s = ct - x^n J_n - \hbar\lambda^n j_n - c\hbar\omega I, \quad n = 1,2,3,\ldots, J_n^2 = I^2 = 1, j_n^2 = -1 \tag{65}$$

such as

$$s^2 = ss^+ = c^2t^2 - x_n x^n + \hbar^2\lambda_n\lambda^n - c^2\hbar^2\omega^2 \tag{66}$$

and gamma factor

$$\gamma^{-1} = \sqrt{1-\frac{v^2}{c^2}\left(1-\hbar^2\frac{d\lambda^n}{\partial x^m}\frac{d\lambda_n}{dx_m}\right)-\left(\hbar\frac{d\omega}{dt}\right)^2} = \frac{d\sqrt{s^2}}{cdt} \tag{67}$$

and where the pseudonorm is bound to the constraints

$$v^2 \leq c^2 \tag{68}$$

$$\frac{dx^n}{d\lambda^n} \geq \hbar \tag{69}$$

$$\frac{dt}{d\omega} \geq \hbar \tag{70}$$

18 de Sitter relativity

dS relativity embeds SR into a 5d space-time. The metric reads:

$$-\eta_{AB}X^AX_B = L_{dS}^2 \tag{71}$$

The momentum reads off

$$g_{\mu\nu}\pi^\mu\pi^\nu = \Omega^2\eta_{\mu\nu}\left(p^\mu p^\nu - \frac{1}{2L_{dS}^2}p^\mu k^\nu + \frac{1}{16L_{dS}^2}k^\mu k^\nu\right) \tag{72}$$

where the normal and conformal momentum are

$$p^\mu = \left(\frac{\varepsilon_p}{c}, \vec{p}\right), p^2 = p_\mu p^\mu = m^2c^2 \tag{73}$$

$$k^\mu = \left(\frac{\varepsilon_k}{c}, \vec{k}\right), k^2 = k_\mu k^\mu = \overline{m}^2c^2 \tag{74}$$

and the modified dispersion relationship in de Sitter relativity can be rewritten as follows

$$\frac{\varepsilon_p^2}{c^2} - p^2 = m^2c^2 + \frac{1}{2L_{dS}^2}\left(\frac{\varepsilon_p\varepsilon_k}{c^2} - \vec{p}\cdot\vec{k} - m\overline{m}c^2 - \frac{1}{8L_{dS}^2}\left(\frac{\varepsilon_k^2}{c^2} - k^2 - \overline{m}^2c^2\right)\right) \tag{75}$$

In dS SR we can derive the following gamma factor with $R = L_{dS}$:

$$\Gamma^{-1} = \sqrt{\left(1 - \frac{\eta_{ij}x^ix^j}{R^2}\right)\left(1 - \eta_{ij}\frac{\dot{x}^i\dot{x}^j}{c^2}\right) + \frac{1}{R^2}\left(2t\eta_{ij}x^i\dot{x}^j - t^2\eta_{ij}\dot{x}^i\dot{x}^j + \frac{(\eta_{ij}x^i\dot{x}^j)^2}{c^2}\right)} \tag{76}$$

19 Uncommon relativities

There are more uncommon relativities out there:

1. Zihua Weng octonionic and sedenionic relativities.
2. Nottale's scale relativity.
3. 3d-time SR by Barashenkov.
4. 3d-time SR by E.A.B. Cole.
5. Multitemporal dS relativity by G. Arcidiacono.
6. Extended tachyonic relativity by E. Recami, Sudarshan, Pavsic and others.
7. Projective 5d SR by Kerner.
8. ...