

Nonlinear Connections on Gerbes, Clifford Modules, and the Index Theorems

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Abstract

The geometry of nonholonomic bundle gerbes, provided with nonlinear connection structure, and related nonholonomic gerbe modules is elaborated as the theory of Clifford modules on nonholonomic manifolds which positively fail to be spin. We explore an approach to such nonholonomic Dirac operators and derive the related Atiyah–Singer index formulas. There are considered certain applications in modern gravity and geometric mechanics of such Clifford–Lagrange/ Finsler gerbes and their realizations as nonholonomic Clifford and Riemann–Cartan modules.

Keywords: Nonholonomic gerbes, nonlinear connections, Riemann–Cartan and Lagrange–Finsler spaces, nonholonomic spin structure, Clifford modules, Dirac operators, the Atiyah–Singer index formulas.

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Contents

1	Introduction	2
2	N–Anholonomic Manifolds	5
2.1	Nonlinear connection structures	5
2.2	Curvatures and torsions of N–anholonomic manifolds	7
3	Examples of N–anholonomic spaces:	11
3.1	Lagrange–Finsler geometry	11
3.2	N–connections and gravity	13
4	Lifs of N–anholonomic Bundle Gerbes and Connections	15
4.1	N–anholonomic bundle gerbes and their lifts	15
4.1.1	Local constructions	15
4.1.2	Global constructions	17
4.2	Curvatures for N–anholonomic bundle gerbe modules	18
5	Nonholonomic Clifford Gerbes and Modules	21
5.1	Clifford d–algebras and N–anholonomic bundles	21
5.2	N–anholonomic Dirac operators and gerbes	23
5.2.1	The index topological formula for holonomic Dirac operators	23
5.2.2	Twisted nonholonomic Dirac operators on Clifford gerbes	24
5.2.3	Main result and concluding remarks	26
A	Local Formulation of N–Connection Geometry	26

1 Introduction

In this paper, we elaborate an approach to the Atiyah–Singer index formulas for manifolds and bundle spaces provided with nonlinear connection structure (in brief, N–connection). We follow the methods developed for bundle gerbes [1] and bundle gerbe modules [2] and related (in certain lines, alternative) results from [3, 4]. It should be noted that bundles and gerbes and their higher generalizations (n –gerbes) can be described both in two equivalent forms: in local geometry, with local functions and forms, and in non–local geometry, by using holonomies and parallel transports, [5, 6, 7], see a review of results in [8].

The nonholonomic structure of a N–anholonomic space is defined by a non–integrable, i.e. nonholonomic (equivalently, anholonomic), distribution

defining a N -connection structure ¹. Nonholonomic geometric configurations are naturally derived in modern gravity and string theories by generic off-diagonal metrics, generalized connections and nonholonomic frame structures [9, 10]. The approach can be elaborated in general form by unifying the concepts of Riemann–Cartan and Finsler–Lagrange spaces [11] and their generalizations on gerbes. It is also related to modelling gravitational field interactions and the Lagrange and/or Hamilton mechanics and further developments to quantum deformations [12, 13], noncommutative geometry and gravity with N -anholonomic structures [14, 15, 16], supermanifolds provided with N -connection structure [17, 18, 19], as well to nonholonomic Lie and Clifford algebroids and their applications in constructing new classes of exact solutions [20].

The general nonholonomic manifolds fail to be spin and there are substantial difficulties in definition of curvature which can be revised and solved in the theory of nonholonomic gerbes. Some of such constructions are relevant to anomalies in quantum field theory [21] when the obstruction to existence of spin structure is regarded as an anomaly in the global definition of spinor fields. The typical solution of this problem is to introduce some additional fields which also have an anomaly in their global definition but choose a such configuration when both anomalies cancel each another.

The failure in existence of usual spin structure is differently treated for the nonholonomic manifolds. At least, for the N -anholonomic spaces, it is possible to define the curvature, which is not a trivial construction for general nonholonomic manifolds, and the so-called N -adapted (nonholonomic) Clifford structures with nontrivial N -connection. This problem was firstly solved for Finsler–Lagrange spaces [22] and their higher order generalizations [23] but it can be also generalized to noncommutative geometry and gravity models with nontrivial nonholonomic structures and Lie–Clifford algebroid symmetries, see reviews and recent results in Refs. [24, 25, 14].

This work is devoted to the index theorems for nonholonomic gerbes and bundle gerbe modules adapted to the N -connection structure. The key idea is to consider the so-called N -anholonomic spin gerbe defined for any N -anholonomic manifold (such a nonholonomic gerbe is a usual spin gerbe for a vanishing N -connection structures and becomes trivial if the basic manifold is spin). We shall construct "twisted" Dirac d -operators ² and investigate their properties for Clifford N -anholonomic modules. Then, we shall define the

¹the rigorous definitions and notations are given below; for our purposes, it is enough to consider nonholonomic spaces defined by N -connections structures (in brief, called N -anholonomic manifolds)

²In brief, we shall write " d -operators and d -objects" for operators and objects distinguished by the N -connection structure, see next section.

Chern character of such of such modules and and show that the usual index formula holds for such a definition but being related to the N-anholonomic structure. The final aim, the proof of index theorems for various types of N-anholonomic spaces, follows from matching up the geometric formalism of Clifford modules and nonholonomic frames with associated N-connections.

The structure of the paper is as follows:

In section 2, we present an introduction into the geometry of N-anholonomic manifolds. The results are formulated in coordinate free form: there are given two equivalent definitions of N-connections, considered basic geometric objects characterizing them and defined and computed in abstract form the torsions and curvatures of N-anholonomic manifolds.

Section 3 is devoted to a study of two explicit examples of N-anholonomic manifolds: the Lagrange-Finsler spaces and Riemann-Cartan manifolds provided with N-connection structures. There are proved two main results: Result 3.1: any model of regular Lagrange mechanics, or Finsler geometry, can be canonically modelled as a N-anholonomic Riemann-Cartan manifold with the basic geometric structures (the N-connection, metric and linear connection) being defined by the fundamental Lagrange, or Finsler, function; Result 3.2: There are N-anholonomic Einstein-Cartan (in particular cases, Einstein) spaces parametrized by nontrivial N-connection structure, nonholonomic frames and, in general, non-Riemann connections defined as generic off-diagonal solutions in modern gravity.

In section 4, there are considered lifting of N-anholonomic bundle gerbes and definition of connection and curvatures on such spaces. We define the (twisted-anholonomic) Chern characters for bundle gerbes and modules induced by N-connections and distinguished metric and linear connection structures. We conclude with two important results/ applications of the theory of nonholonomic gerbes: Result 4.1: Any regular Lagrange (Finsler) configuration is topologically characterized by its Chern character computed from canonical connections defined by Lagrangian (fundamental Finsler function). Result 4.2: the geometric constructions for a N-anholonomic Riemann-Cartan manifold (including exact solutions in gravity with nontrivial N-connection structure) can be globalized to N-anholonomic gerbe configurations and characterized by the corresponding Chern character.

Section 5 presents the Main result of this paper: The twisted index formula for N-anholonomic Dirac operators and related gerbe constructions (see Theorem 5.2). We introduce Clifford d-algebras on N-anholonomic bundles and define twisted nonholonomic Dirac operators on Clifford gerbes. We conclude that there are certain fundamental topological characteristics derived from a regular fundamental Lagrange, or Finsler, function and that

such indices classify new classes of exact solutions in gravity globalized on gravitational gerbe configurations.

The Appendix contains a set of component formulas for N–connections, metric and linear connection structures and related torsions and curvatures on N–anholonomic manifolds. They may be considered for some local proofs of results in the main part of the paper, as well for an introduction and some applications in modern physics.

2 N–Anholonomic Manifolds

We formulate a coordinate free approach to the geometry of nonholonomic manifolds. The reader may consult details in Refs. [9, 10, 11, 13]. Some important component/coordinate formulas are given in the Appendix.

2.1 Nonlinear connection structures

Let \mathbf{V} be a smooth manifold of dimension $(n + m)$ with local splitting in any point $u \in \mathbf{V}$ of type $\mathbf{V}_u = M_u \oplus V_u$, where M is a n –dimensional subspace and V is a m –dimensional subspace. It is supposed that one exists such a local decomposition when $\mathbf{V} \rightarrow M$ is a surjective submersion. Two important particular cases are that of a vector bundle, when we shall write $\mathbf{V} = \mathbf{E}$ (with \mathbf{E} being the total space of a vector bundle $\pi : \mathbf{E} \rightarrow M$ with the base space M) and that of tangent bundle when we shall consider $\mathbf{V} = \mathbf{TM}$. The differential of a map $\pi : \mathbf{V} \rightarrow M$ defined by fiber preserving morphisms of the tangent bundles $T\mathbf{V}$ and TM is denoted by $\pi^\top : T\mathbf{V} \rightarrow TM$. The kernel of π^\top defines the vertical subspace $v\mathbf{V}$ with a related inclusion mapping $i : v\mathbf{V} \rightarrow T\mathbf{V}$.

Definition 2.1 *A nonlinear connection (N–connection) \mathbf{N} on a manifold \mathbf{V} is defined by the splitting on the left of an exact sequence*

$$0 \rightarrow v\mathbf{V} \xrightarrow{i} T\mathbf{V} \rightarrow T\mathbf{V}/v\mathbf{V} \rightarrow 0, \quad (1)$$

i. e. by a morphism of submanifolds $\mathbf{N} : T\mathbf{V} \rightarrow v\mathbf{V}$ such that $\mathbf{N} \circ i$ is the unity in $v\mathbf{V}$.

The exact sequence (1) states a nonintegrable (nonholonomic, equivalently, anholonomic) distribution on \mathbf{V} , i.e. this manifold is nonholonomic. We can say that a N–connection is defined by a global splitting into conventional horizontal (h) subspace, $(h\mathbf{V})$, and vertical (v) subspace, $(v\mathbf{V})$, corresponding to the Whitney sum

$$T\mathbf{V} = h\mathbf{V} \oplus_N v\mathbf{V} \quad (2)$$

where $h\mathbf{V}$ is isomorphic to M . We put the label N to the \oplus in order to emphasize that such a splitting is associated to a N -connection structure. In this paper, we shall avoid local coordinate considerations. For convenience, in Appendix, we give some important local formulas (see, for instance, the local representation for a N -connection (21)) for the basic geometric objects and formulas on spaces provided with N -connection structure. Here, we note that the concept of N -connection came from E. Cartan's works on Finsler geometry [27] (see a detailed historical study in Refs. [28, 13, 14] and alternative approaches like the Ehressmann connection [29, 30]).

Any manifold admitting an exact sequence of type (1) admits an N -connection structure. If $\mathbf{V} = \mathbf{E}$, a N -connection exists for any vector bundle \mathbf{E} over a paracompact manifold M , see proof in Ref. [28].

The geometric objects on spaces provided with N -connection structure are denoted by "bolfaced" symbols. Such objects may be defined in N -adapted form to the h - and v -decomposition (2). Following conventions from [28, 22, 24, 14], one call such objects to be d -objects (i. e. they are distinguished by the N -connection; one considers d -vectors, d -forms, d -tensors, d -spinors, d -connections, ...). For instance, a d -vector is an element \mathbf{X} of the module of the vector fields $\chi(\mathbf{V})$ on \mathbf{V} , which in N -adapted form may be written

$$\mathbf{X} = h\mathbf{X} + v\mathbf{X} \text{ or } \mathbf{X} = X \oplus_N \bullet X,$$

where $h\mathbf{X}$ (equivalently, X) is the h -component and $v\mathbf{X}$ (equivalently, $\bullet X$) is the v -component of \mathbf{X} .

A N -connection is characterized by its **N -connection curvature** (the Nijenhuis tensor)

$$\Omega(\mathbf{X}, \mathbf{Y}) \doteq [\bullet X, \bullet Y] + \bullet[\mathbf{X}, \mathbf{Y}] - \bullet[\bullet X, \mathbf{Y}] - \bullet[\mathbf{X}, \bullet Y] \quad (3)$$

for any $\mathbf{X}, \mathbf{Y} \in \chi(\mathbf{V})$, where $[\mathbf{X}, \mathbf{Y}] \doteq \mathbf{X}\mathbf{Y} - \mathbf{Y}\mathbf{X}$ and $\bullet[\cdot, \cdot]$ is the v -projection of $[\cdot, \cdot]$, see also the coordinate formula (22) in Appendix. This d -object Ω was introduced in Ref. [31] in order to define the curvature of a nonlinear connection in the tangent bundle over a smooth manifold. But this can be extended for any nonholonomic manifold, nonholnomic Clifford structure and any noncommutative / supersymmetric versions of bundle spaces provided with N -connection structure, i. e. with nonintegrable distributions of type (2), see [13, 14, 25].

Proposition 2.1 *A N -connection structure on \mathbf{V} defines a nonholonomic N -adapted frame (vielbein) structure $\mathbf{e} = (e, \bullet e)$ and its dual $\tilde{\mathbf{e}} = (\tilde{e}, \bullet \tilde{e})$ with e and $\bullet \tilde{e}$ linearly depending on N -connection coefficients.*

Proof. It follows from explicit local constructions, see formulas (24), (23) and (25) in Appendix. \square

Definition 2.2 *A manifold \mathbf{V} is called N -anholonomic if it is defined a local (in general, nonintegrable) distribution (2) on its tangent space $T\mathbf{V}$, i.e. \mathbf{V} is N -anholonomic if it is enabled with a N -connection structure (1).*

All spinor and gerbe constructions in this paper will be performed for N -anholonomic manifolds.

2.2 Curvatures and torsions of N -anholonomic manifolds

One can be defined N -adapted linear connection and metric structures on \mathbf{V} :

Definition 2.3 *A distinguished connection (d -connection) \mathbf{D} on a N -anholonomic manifold \mathbf{V} is a linear connection conserving under parallelism the Whitney sum (2). For any $\mathbf{X} \in \chi(\mathbf{V})$, one have a decomposition into h - and v -covariant derivatives,*

$$\mathbf{D}_{\mathbf{X}} \doteq \mathbf{X} \rfloor \mathbf{D} = X \rfloor \mathbf{D} + \bullet X \rfloor \mathbf{D} = D_X + \bullet D_X. \quad (4)$$

The symbol " \rfloor " in (4) denotes the interior product. We shall write conventionally that $\mathbf{D} = (D, \bullet D)$.

For any d -connection \mathbf{D} on a N -anholonomic manifold \mathbf{V} , it is possible to define the curvature and torsion tensor in usual form but adapted to the Whitney sum (2):

Definition 2.4 *The torsion*

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{D}_{\mathbf{X}} \mathbf{Y} - \mathbf{D}_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] \quad (5)$$

of a d -connection $\mathbf{D} = (D, \bullet D)$, for any $\mathbf{X}, \mathbf{Y} \in \chi(\mathbf{V})$, has a N -adapted decomposition

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) = \mathbf{T}(X, Y) + \mathbf{T}(X, \bullet Y) + \mathbf{T}(\bullet X, Y) + \mathbf{T}(\bullet X, \bullet Y). \quad (6)$$

By further h - and v -projections of (6), denoting $h\mathbf{T} \doteq T$ and $v\mathbf{T} \doteq \bullet T$, taking in the account that $h[\bullet X, \bullet Y] = 0$, one proves

Theorem 2.1 *The torsion of a d-connection $\mathbf{D}=(D, \bullet D)$ is defined by five nontrivial d-torsion fields adapted to the h- and v-splitting by the N-connection structure*

$$\begin{aligned} T(X, Y) &\doteq D_X Y - D_Y X - h[X, Y], \quad \bullet T(X, Y) \doteq \bullet[Y, X], \\ T(X, \bullet Y) &\doteq -\bullet D_Y X - h[X, \bullet Y], \quad \bullet T(X, \bullet Y) \doteq \bullet D_X Y - \bullet[X, \bullet Y], \\ \bullet T(\bullet X, \bullet Y) &\doteq \bullet D_X \bullet Y - \bullet D_Y \bullet X - \bullet[\bullet X, \bullet Y]. \end{aligned}$$

The d-torsions $T(X, Y)$, $\bullet T(\bullet X, \bullet Y)$ are called respectively the $h(hh)$ -torsion, $v(vv)$ -torsion and so on. The formulas (34) in Appendix present a local proof of this Theorem.

Definition 2.5 *The curvature of a d-connection $\mathbf{D}=(D, \bullet D)$ is defined*

$$\mathbf{R}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{D}_X \mathbf{D}_Y - \mathbf{D}_Y \mathbf{D}_X - \mathbf{D}_{[X, Y]} \quad (7)$$

for any $\mathbf{X}, \mathbf{Y} \in \chi(\mathbf{V})$.

Denoting $h\mathbf{R} = R$ and $v\mathbf{R} = \bullet R$, by straightforward calculations, one check the properties

$$\begin{aligned} R(\mathbf{X}, \mathbf{Y}) \bullet Z &= 0, \quad \bullet R(\mathbf{X}, \mathbf{Y}) Z = 0, \\ \mathbf{R}(\mathbf{X}, \mathbf{Y}) \mathbf{Z} &= R(\mathbf{X}, \mathbf{Y}) Z + \bullet R(\mathbf{X}, \mathbf{Y}) \bullet Z \end{aligned}$$

for any for any $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \chi(\mathbf{V})$.

Theorem 2.2 *The curvature \mathbf{R} of a d-connection $\mathbf{D}=(D, \bullet D)$ is completely defined by six d-curvatures*

$$\begin{aligned} \mathbf{R}(X, Y) Z &= (D_X D_Y - D_Y D_X - D_{[X, Y]} - \bullet D_{[X, Y]}) Z, \\ \mathbf{R}(X, Y) \bullet Z &= (D_X D_Y - D_Y D_X - D_{[X, Y]} - \bullet D_{[X, Y]}) \bullet Z, \\ \mathbf{R}(\bullet X, Y) Z &= (\bullet D_X D_Y - D_Y \bullet D_X - D_{[\bullet X, Y]} - \bullet D_{[\bullet X, Y]}) Z, \\ \mathbf{R}(\bullet X, Y) \bullet Z &= (\bullet D_X \bullet D_Y - \bullet D_Y \bullet D_X - D_{[\bullet X, Y]} - \bullet D_{[\bullet X, Y]}) \bullet Z, \\ \mathbf{R}(\bullet X, \bullet Y) Z &= (\bullet D_X D_Y - D_Y \bullet D_X - \bullet D_{[\bullet X, \bullet Y]}) Z, \\ \mathbf{R}(\bullet X, \bullet Y) \bullet Z &= (\bullet D_X D_Y - D_Y \bullet D_X - \bullet D_{[\bullet X, \bullet Y]}) \bullet Z. \end{aligned}$$

The proof of Theorems 2.1 and 2.2 is given for vector bundles provided with N-connection structure in Ref. [28]. Similar Theorems and respective proofs hold true for superbundles [17], for noncommutative projective modules [14] and for N-anholonomic metric-affine spaces [11], where there are also give the main formulas in abstract coordinate form. The formulas (39) from Appendix consist a coordinate proof of Theorem 2.2.

Definition 2.6 A metric structure \check{g} on a N -anholonomic space \mathbf{V} is a symmetric covariant second rank tensor field which is not degenerated and of constant signature in any point $\mathbf{u} \in \mathbf{V}$.

In general, a metric structure is not adapted to a N -connection structure.

Definition 2.7 A d -metric $\mathbf{g} = g \oplus_N \bullet g$ is a usual metric tensor which contracted to a d -vector results in dual d -vector, d -covector (the duality being defined by the inverse of this metric tensor).

The relation between arbitrary metric structures and d -metrics is established by

Theorem 2.3 Any metric \check{g} can be equivalently transformed into a d -metric

$$\mathbf{g} = g(X, Y) + \bullet g(\bullet X, \bullet Y) \quad (8)$$

for a corresponding N -connection structure.

Proof. We introduce denotations $h\check{g}(X, Y) \doteq g(X, Y)$ and $v\check{g}(\bullet X, \bullet Y) = \bullet g(\bullet X, \bullet Y)$ and try to find a N -connection when

$$\check{g}(X, \bullet Y) = 0 \quad (9)$$

for any $\mathbf{X}, \mathbf{Y} \in \chi(\mathbf{V})$. In local form, the equation (9) is just an algebraic equation for $\mathbf{N} = \{N_i^a\}$, see formulas (26), (27) and (28) and related explanations in Appendix. \square

Definition 2.8 A d -connection \mathbf{D} on \mathbf{V} is said to be metric (i.e. to satisfy the metric compatibility, metricity, conditions) with a metric \check{g} and its equivalent d -metric \mathbf{g} if there are satisfied the conditions

$$\mathbf{D}_X \mathbf{g} = \mathbf{0}. \quad (10)$$

By explicit h - and v -projecting of (10) one prove

Proposition 2.2 A d -connection \mathbf{D} on \mathbf{V} is metric if and only if

$$D_X g = 0, D_X \bullet g = 0, \bullet D_X g = 0, \bullet D_X \bullet g = 0.$$

One holds this important

Conclusion 2.1 Following Propositions 2.1 and 2.2 and Theorem 2.3, we can elaborate the geometric constructions on a N -anholonomic manifold \mathbf{V} in N -adapted form by considering N -adapted frames $\mathbf{e} = (e, \bullet e)$ and co-frames $\tilde{\mathbf{e}} = (\tilde{e}, \bullet \tilde{e})$, d -connection \mathbf{D} and d -metric fields $\mathbf{g} = [g, \bullet g]$.

In Riemannian geometry, there is a preferred linear Levi–Civita connection ∇ which is torsionless, i.e.

$$\nabla \mathbf{T}(\mathbf{X}, \mathbf{Y}) \doteq \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0$$

and defined by the metric structure. On a general N -anholonomic manifold \mathbf{V} provided with a d -metric structure $\mathbf{g} = [g, \bullet g]$, the Levi–Civita connection defined by this metric is not adapted to the N -connection, i. e. to the splitting (2). The h - and v -distributions are nonintegrable ones and any d -connection adapted to a such splitting contains nontrivial d -torsion coefficients. Nevertheless, one exists a minimal extension of the Levi–Civita connection to a canonical d -connection which is defined only by a metric \check{g} .

Theorem 2.4 *For any d -metric $\mathbf{g} = [g, \bullet g]$ on a N -anholonomic manifold \mathbf{V} , there is a unique metric canonical d -connection $\widehat{\mathbf{D}}$ satisfying the conditions $\widehat{\mathbf{D}}\mathbf{g} = 0$ and vanishing $h(hh)$ -torsion, $v(vv)$ -torsion, i. e. $\widehat{T}(X, Y) = 0$ and $\bullet \widehat{T}(\bullet X, \bullet Y) = 0$.*

Proof. The formulas (35) and (37) and related discussions in Appendix give a proof, in component form, of this Theorem. \square

The following proposition gathers some basic information about N -anholonomic manifolds.

Corollary 2.1 *A N -connection structure defines three important geometric objects:*

1. *a (pseudo) Euclidean N -metric structure $\eta = \eta \oplus_N \bullet \eta$, i.e. a d -metric with (pseudo) Euclidean metric coefficients with respect to $\tilde{\mathbf{e}}$ defined only by \mathbf{N} ;*
2. *a metric canonical d_N -connection $\widehat{\mathbf{D}}^N$ defined only by η and \mathbf{N} ;*
3. *a nonmetric Berwald type linear connection \mathbf{D}^B .*

Proof. Fixing a signature for the metric, $sign\eta = (\pm, \pm, \dots, \pm)$, we introduce these values in (28) we get $\eta = \eta \oplus_N \bullet \eta$ of type (8), i.e. we prove the point 1. The point 2 is to be proved by an explicit construction by considering the coefficients of η into (37). This way we get a canonical d -connection induced by the N -connection coefficients and satisfying the metricity conditions (10). In modern literature [32], one emphasizes constructions derived for the so-called Berwald type d -connection \mathbf{D}^B , the "most" minimal (linear on Ω) extension of the Levi–Civita connection, see formulas (38). Such d -connections can be defined for an arbitrary d -metric $\mathbf{g} = [g, \bullet g]$, or for

any $\eta = \eta \oplus_N \bullet\eta$. They are only "partially" metric because, for instance, $D^B g = 0$ and $\bullet D^B \bullet g = 0$ but, in general, $D^B \bullet g \neq 0$ and $\bullet D^B g \neq 0$, i. e. $D^B \mathbf{g} \neq 0$, see Proposition 2.2. \square

Remark 2.1 *The geometrical objects $\eta, \widehat{\mathbf{D}}^N, \mathbf{D}^B$ for η , nonholonomic bases $\mathbf{e} = (e, \bullet e)$ and $\tilde{\mathbf{e}} = (\tilde{e}, \bullet \tilde{e})$, see Proposition 2.1 and the N-connection curvature Ω (3) define completely the main properties of an N-anholonomic manifold \mathbf{V} . It is possible to extend the constructions for any additional d-metric and canonical d-connection structures. For our further considerations on nonholonomic Clifford/spinor structures the class of metric d-connections plays a preferred role. That why we emphasize the physical importance of d-connections $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{D}}^N$ instead of \mathbf{D}^B or any nonmetric d-connections.*

Finally, in this section, we note that on N-anholonomic manifolds d-torsions and d-curvatures can be computed for any type of d-connection structure, see Theorems 2.1 and 2.2 and the component formulas (34) and (39).

3 Examples of N-anholonomic spaces:

For certain parametrizations of N-connection, d-metric and d-connection coefficients of a N-anholonomic space, it is possible to model various classes of (generalized) Lagrange, Finsler and Riemann-Cartan spaces. We briefly analyze three such nonholonomic geometric structures.

3.1 Lagrange-Finsler geometry

Such geometries are usually defined on tangent bundles [28] but it is possible to model such structures on general N-anholonomic manifolds, in particular in (pseudo) Riemannian and Riemann-Cartan geometry if nonholonomic frames are introduced into consideration [9, 10, 11, 20]. Let us outline the first approach when the N-anholonomic manifold \mathbf{V} is just a tangent bundle (TM, π, M) , where M is a n -dimensional base manifold, π is a surjective projection and TM is the total space. One denotes by $\widetilde{TM} = TM \setminus \{0\}$ where $\{0\}$ means the null section of map π .

We consider a differentiable fundamental Lagrange function $L(x, y)$ defined by a map $L : (x, y) \in TM \rightarrow L(x, y) \in \mathbb{R}$ of class \mathcal{C}^∞ on \widetilde{TM} and continuous on the null section $0 : M \rightarrow TM$ of π . The values $x = \{x^i\}$ are local coordinates on M and $(x, y) = (x^i, y^k)$ are local coordinates on TM .

For simplicity, we consider this Lagrangian to be regular, i.e. with nondegenerated Hessian

$${}^L g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} \quad (11)$$

when $\text{rank} |g_{ij}| = n$ on \widetilde{TM} and the left up "L" is an abstract label pointing that the values are defined by the Lagrangian L .

Definition 3.1 *A Lagrange space is a pair $L^n = [M, L(x, y)]$ with ${}^L g_{ij}(x, y)$ being of constant signature over \widetilde{TM} .*

The notion of Lagrange space was introduced by J. Kern [33] and elaborated in details in Ref. [28] as a natural extension of Finsler geometry.

Theorem 3.1 *There are canonical N-connection ${}^L \mathbf{N}$, almost complex ${}^L \mathbf{F}$, d-metric ${}^L \mathbf{g}$ and d-connection ${}^L \widehat{\mathbf{D}}$ structures defined by a regular Lagrangian $L(x, y)$ and its Hessian ${}^L g_{ij}(x, y)$ (11).*

Proof. The canonical ${}^L \mathbf{N}$ is defined by certain nonlinear spray configurations related to the solutions of Euler–Lagrange equations, see the local formula (43) in Appendix. There it is given the explicit matrix representation of ${}^L \mathbf{F}$ (44) which is a usual definition of almost complex structure, after ${}^L \mathbf{N}$ and N-adapted bases have been constructed. The d-metric (45) is a local formula for ${}^L \mathbf{g}$. Finally, the canonical d-connection ${}^L \widehat{\mathbf{D}}$ is a usual one but for ${}^L \mathbf{g}$ and ${}^L \mathbf{N}$ on \widetilde{TM} . \square

A similar Theorem can be formulated and proved for Finsler geometry:

Remark 3.1 *A Finsler space is defined by a fundamental Finsler function $F(x, y)$, being homogeneous of type $F(x, \lambda y) = \lambda F(x, y)$, for nonzero $\lambda \in \mathbb{R}$, may be considered as a particular case of Lagrange geometry when $L = F^2$.*

From the Theorem 3.1 and Remark 3.1 one follows:

Result 3.1 *Any Lagrange mechanics with regular Lagrangian $L(x, y)$ (any Finsler geometry with fundamental function $F(x, y)$) can be modelled as a nonholonomic Riemann–Cartan geometry with canonical structures ${}^L \mathbf{N}$, ${}^L \mathbf{g}$ and ${}^L \widehat{\mathbf{D}}$ (${}^F \mathbf{N}$, ${}^F \mathbf{g}$ and ${}^F \widehat{\mathbf{D}}$) defined on an N-anholonomic manifold \mathbf{V} .*

It was concluded that any regular Lagrange mechanics/Finsler geometry can be geometrized/modelled as an almost Kähler space with canonical N-connection distribution, see [28], and for N-anholonomic Fedosov manifolds [13]. Such approaches based on almost complex structures are related with

standard symplectic approaches to classical mechanics and field theory, for a review of results see Ref. [30].

For applications in optics of nonhomogeneous media [28] and gravity (see, for instance, Refs. [9, 10, 11, 26, 20]) one considers metrics of type $g_{ij} \sim e^{\lambda(x,y)} L g_{ij}(x, y)$ which can not be derived from a mechanical Lagrangian but from an effective "energy" function. In the so-called generalized Lagrange geometry one considers Sasaki type metrics (45), see the Appendix, with any general coefficients both for the metric and N-connection.

3.2 N-connections and gravity

Now we show how N-anholonomic configurations can be defined in gravity theories. In this case, it is convenient to work on a general manifold \mathbf{V} , $\dim \mathbf{V} = n + m$ enabled with a global N-connection structure, instead of the tangent bundle \widetilde{TM} .

For N-connection splittings of (pseudo) Riemannian-Cartan spaces of dimension $(n + m)$ (there were also considered (pseudo) Riemannian configurations), the Lagrange and Finsler type geometries were modelled by N-anholonomic structures as exact solutions of gravitational field equations [11, 10, 16]. Inversely, modern approaches to (super) string gravity models deal with nontrivial torsion and (super) vielbein fields which under corresponding parametrizations model N-anholonomic spaces [17, 23, 25]. We summarize here some geometric properties of gravitational models with nontrivial N-anholonomic structure.

Definition 3.2 *A N-anholonomic Riemann-Cartan manifold ${}^{RC}\mathbf{V}$ is defined by a d-metric \mathbf{g} and arbitrary metric d-connection \mathbf{D} structures adapted an exact splitting (1) defined on this manifold.*

Equivalently, ${}^{RC}\mathbf{V}$ is a Riemann-Cartan manifold provided with N-connection structure defining a global nonintegrable splitting of type (2). The d-metric structure \mathbf{g} on ${}^{RC}\mathbf{V}$ is of type (8) and satisfies the metricity conditions (10). With respect to a local coordinate basis \mathbf{g} is parametrized by a generic off-diagonal metric ansatz (27), see Appendix. In a particular case we can take $\mathbf{D} = \widehat{\mathbf{D}}$ and treat the torsion $\widehat{\mathbf{T}}$ as a nonholonomic frame effect induced by nonintegrable N-splitting. For more general applications, we have to consider additional torsion components, for instance, by the so-called H -field in string gravity.

Let us denote by $Ric(\mathbf{D})$ and $Sc(\mathbf{D})$, respectively, the Ricci tensor and curvature scalar defined by any metric d-connection \mathbf{D} and d-metric \mathbf{g} on

${}^{RC}\mathbf{V}$, see also the component formulas (40), (41) and (42) in Appendix. The Einstein equations are

$$En(\mathbf{D}) \doteq Ric(\mathbf{D}) - \frac{1}{2}\mathbf{g}Sc(\mathbf{D}) = \Upsilon \quad (12)$$

where the source Υ reflects any contributions of matter fields and corrections from, for instance, string/brane theories of gravity. In a closed physical model, the equation (12) have to be completed with moving equations for the matter fields, torsion contributions and so on (for instance, in the Einstein–Cartan theory one considers algebraic equations for the torsion and its source)... It should be noted here that because of nonholonomic structure of ${}^{RC}\mathbf{V}$, the tensor $Ric(\mathbf{D})$ is not symmetric and that $\mathbf{D}[En(\mathbf{D})] \neq 0$ which imposes a more sophisticate form of conservation laws on such spaces with generic "local anisotropy", see discussion in [11, 24] (this is a similar situation like when nonholonomic constraints in Lagrange mechanics modifies the definition of conservation laws). A very important class of models can be elaborated when $\Upsilon = diag[\lambda^h(\mathbf{u})g, \lambda^v(\mathbf{u}) \bullet g]$, which defines the so-called N-anholonomic Einstein spaces.

Result 3.2 *Various classes of vacuum and nonvacuum exact solutions of (12) parametrized by generic off-diagonal metrics, noholonomic vielbeins and Levi-Civita or non-Riemannian connections in Einstein and extra dimension gravity models define N-anholonomic Einstein–Cartan (in particular, Einstein) spaces.*

In explicit form such exact solutions (for instance, with noncommutative, algebroid, toroidal, ellipsoid, ... symmetries) have been constructed in Refs. [9, 10, 13, 14, 16, 20, 26, 11, 24].

Remark 3.2 *A subclass of N-anholonomic Einstein spaces were related to generic off-diagonal solutions in general relativity by such nonholonomic constraints when $Ric(\hat{\mathbf{D}}) = Ric(\nabla)$ even $\hat{\mathbf{D}} \neq \nabla$, where $\hat{\mathbf{D}}$ is the canonical d-connection and ∇ is the Levi-Civita connection, see formulas (35) and (36) in Appendix and details in Ref. [20].*

A new direction in modern gravity is related to analogous gravity models when certain gravitational effects and, for instance, black hole configurations are modelled by optical and acoustic media, see a recent review or results in [34]. Following our approach on geometric unification of gravity and geometric mechanics in terms of N-anholonomic spaces, one holds

Theorem 3.2 *A Lagrange (Finsler) space can be canonically modelled as an exact solution of the Einstein equations (12) on a N-anholonomic Riemann–Cartan space if and if the canonical N-connection ${}^L\mathbf{N}$ (${}^F\mathbf{N}$), d-metric ${}^L\mathbf{g}$ (${}^F\mathbf{g}$) and d-connection ${}^L\widehat{\mathbf{D}}$ (${}^F\widehat{\mathbf{D}}$) structures defined by the corresponding fundamental Lagrange function $L(\mathbf{x}, \mathbf{y})$ (Finsler function $F(\mathbf{x}, \mathbf{y})$) satisfy the gravitational field equations for certain physically reasonable sources Υ .*

Proof. We sketch the idea: It can be performed in local form by considering the Einstein tensor (42) defined by the ${}^L\mathbf{N}$ (${}^F\mathbf{N}$) in the form (43) and ${}^L\mathbf{g}$ (${}^F\mathbf{g}$) in the form (45) inducing the canonical d-connection ${}^L\widehat{\mathbf{D}}$ (${}^F\widehat{\mathbf{D}}$). For certain zero or nonzero Υ , such N-anholonomic configurations may be defined by exact solutions of the Einstein equations for a d-connection structure. A number of explicit examples were constructed for N-anholonomic Einstein spaces [9, 10, 13, 14, 16, 20, 26, 11, 24].□

It should be noted that Theorem states explicit conditions when the Result 3.1 holds for N-anholonomic Einstein spaces.

Conclusion 3.1 *Generic off-diagonal metric and vielbein structures in gravity and regular Lagrange mechanics models can be geometrized in a unified form on N-anholonomic manifolds. In general, such spaces are not spin and this presents a strong motivation for elaborating the theory of nonholonomic gerbes and related Clifford/spinor structures developed in this work.*

4 Lifts of N-anholonomic Bundle Gerbes and Connections

In this section we present an introduction into the geometry of lifts of nonholonomic bundle gerbes and related N-anholonomic modules. We define connections and curvatures for such bundle modules. This material reproduce in the corresponding holonomic limits certain fundamental results from [1, 2, 4, 3].

4.1 N-anholonomic bundle gerbes and their lifts

4.1.1 Local constructions

On N-anholonomic manifolds, one deals with nonintegrable h- and v-splittings of geometric objects, described by the so-called d-objects (for instance, d-vectors, d-spinors, d-tensors, d-connections, ... like we considered in the previous section). It is convenient to introduce the concept of Lie

d-group $\mathbf{G} = (G, \bullet G)$ [22, 23, 14, 24, 25, 26] which is just a couple of two usual Lie groups G and $\bullet G$ associated to a N-connection splitting (2). We conventionally consider a central extension of a finite dimensional of Lie d-groups \mathbf{G} to $\check{\mathbf{G}}$, defined by a map $\pi : \check{\mathbf{G}} \rightarrow \mathbf{G}$ such that it is defined the exact sequence

$$0 \rightarrow \mathbb{Z}_k \rightarrow \check{\mathbf{G}} \rightarrow \mathbf{G} \rightarrow 1 \quad (13)$$

where $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ denotes the cyclic subgroup of the circle $U(1)$. This sequence of d-groups splits into respective horizontal component

$$0 \rightarrow \mathbb{Z}_k \rightarrow \check{G} \rightarrow G \rightarrow 1$$

and vertical component

$$0 \rightarrow \mathbb{Z}_k \rightarrow \bullet \check{G} \rightarrow \bullet G \rightarrow 1.$$

Let us denote by \mathbf{U} and $\check{\mathbf{U}}$ the corresponding right principal sets: the are just \mathbf{G} and $\check{\mathbf{G}}$ but conventionally re-defined in order to consider distinguished (not mixing the h- and v-subsets) actions of \mathbf{G} on \mathbf{U} and $\check{\mathbf{G}}$ on $\check{\mathbf{U}}$. We consider an equivariant $\check{\mathbf{G}}$ bundle $\mathbf{v}_{\mathbf{U}} \doteq \mathbf{U} \times \mathbf{v}$, where \mathbf{v} is a d-vector space and a finite-dimensional representation $\rho : \check{\mathbf{G}} \rightarrow GL(\mathbf{v})$ with the $\check{\mathbf{G}}$ action

$$\check{\mathbf{g}}(\mathbf{u}, \mathbf{v}) = (\mathbf{u}\check{\mathbf{g}}^{-1}, \rho(\check{\mathbf{g}})\mathbf{v}) = \left\{ \begin{array}{l} (x\check{g}^{-1}, \rho(\check{g})v), \\ (y \bullet \check{g}^{-1}, \bullet \rho(\bullet \check{g}) \bullet v) \end{array} \right\}.$$

The pull-back of the \mathbb{Z}_k distinguished bundle $\check{\mathbf{G}} \rightarrow \mathbf{G}$ is

$$\Psi = \tau^* \check{\mathbf{G}} \rightarrow \mathbf{U} \times \mathbf{U}$$

defined by $\Psi_{(\mathbf{u}_1, \mathbf{u}_2)} \doteq \{\pi(\check{\mathbf{g}}) = \tau(\mathbf{u}_1, \mathbf{u}_2)\}$ for any $\check{\mathbf{g}} \in \check{\mathbf{G}}$, where $\tau : \mathbf{U} \times \mathbf{U} \rightarrow \mathbf{G}$ is a canonical map $\mathbf{u}_1 \tau(\mathbf{u}_1, \mathbf{u}_2) \rightarrow \mathbf{u}_2$ translating \mathbf{u}_1 into \mathbf{u}_2 . So, $\Psi_{(\mathbf{u}_1, \mathbf{u}_2)}$ is the set of all distinguished lifts of $\tau(\mathbf{u}_1, \mathbf{u}_2)$ to $\check{\mathbf{G}}$ and Ψ is the \mathbb{Z}_k -principal bundle provided with a trivial N-connection (in this case, with zero N-connection curvature). The bundle Ψ has a module the $\check{\mathbf{G}}$ -equivariant bundle $\mathbf{v}_{\mathbf{U}} \rightarrow \mathbf{U}$. This follows from the fact that for any two pull-backs \mathbf{U}_1 and \mathbf{U}_2 of $\mathbf{v}_{\mathbf{U}}$ as two respective projections $\mathbf{U} \times \mathbf{U} \rightarrow \mathbf{U}$ one has that $\Psi_{(\mathbf{u}_1, \mathbf{u}_2)} \subset \check{\mathbf{G}}$ transforms the distinguished fiber in $(\mathbf{u}_1, \mathbf{u}_2)$ of \mathbf{U}_1 into the corresponding one of \mathbf{U}_2 related by the representation map ρ . Having also the $\check{\mathbf{G}}$ -equivariance, of $End(\mathbf{v}_{\mathbf{U}})$, we can write $End(\mathbf{v}_{\mathbf{U}})/\mathbf{G} = \mathbf{End}(\mathbf{v})$ for distinguished endomorphisms.

4.1.2 Global constructions

The above presented constructions can be globalized to the case of N -anholonomic manifold \mathbf{V} instead of the d -vector space \mathbf{v} (the d -objects with trivial splitting can be considered for any point of \mathbf{V}). The porcedure is completely similar to that given for "holonomic" manifolds in [4] but it should be performed in a form to preserve the N -connection splitting (2). This may be achieved by applied globalizing the bundle Ψ and transforming it into a nonholonomic bundle.

Having in mind the distinguished extension (13), we replace the set \mathbf{U} by a principal N -anholonomic \mathbf{G} -bundle $\pi : \mathbf{B} \rightarrow \mathbf{V}$ and consider the product $\mathbf{B} \times \mathbf{B} \rightarrow \mathbf{V}$ instead of $\mathbf{U} \times \mathbf{U}$. Like for a trivial point of \mathbf{U} , the globalized map $\tau : \mathbf{B} \times \mathbf{B} \rightarrow \mathbf{G}$ allows to introduce $\Psi = \tau^* \check{\mathbf{G}}$ being the \mathbb{Z}_k -bundle over $\mathbf{B} \times \mathbf{B}$ which defines a lifting N -anholonomic bundle gerbe if to follow the terminology for holonomic constructions, [1].

We can consider d -tensor objects of weight q as Ψ^q -modules being non-holonomic variants of bundle gerbe modules for the N -anholonomic bundle gerbe $\Psi^q \doteq \Psi^{\otimes q}$. In more explicit form, we use a $\check{\mathbf{G}}$ -equivariant bundle $\mathbf{W} \rightarrow \mathbf{B}$ for the action of $\check{\mathbf{G}}$ of \mathbf{B} :

Definition 4.1 *The N -anholonomic $\check{\mathbf{G}}$ -equivariant bundle $\mathbf{W} \rightarrow \mathbf{B}$ with defined action of weight q of the isotropy distinguished subgroups states \mathbf{W} as a Ψ^q -module.*

The space \mathbf{W} can be also treated as a vector bundle direct sum of Ψ^q -modules, all adapted to the N -connection structure, i.e. preserving the h - and v -decomposition by (2). This allows us to concentrate our attention only to "boldfaced" Ψ^q -modules carrying out all information about nonholonomic and non-trivial topological configurations. Such constructions run parallel to the usual theory of vector bundles provided with N -connection structure and in a more formalized form (unifying the approaches to gauge fields, gravity and geometrized mechanics) to N -anholonomic manifolds.

It should be noted that if \mathbf{W} is a Ψ -module, then we get a trivial module but it can provided with a nontrivial N -connection (with nonvanishing N -connection curvature). In such cases, one works with constructions of type $\mathcal{E}(\mathbf{W}) = \text{End}(\mathbf{W})/\mathbf{G}$ splitting into h - and v -subspaces and this allow us to reformulate in nonholonomic form, for N -anholonomic Ψ -modules, the main properties of such spaces formally formulated for trivial N -connection structure [35].

Proposition 4.1 *The Ψ^q -modules satisfy the following N -adapted properties:*

1. N -anholonomic Ψ^q -modules and bundles on \mathbf{V} are bijective equivalent.
2. The bundle of N -adapted endomorphisms of a Ψ^q -module is a Ψ^0 -module.
3. The direct sum of two Ψ^q -modules is a Ψ^q -module.
4. The d -tensor product of a Ψ^{q_1} -module to a Ψ^{q_2} -module results in a $\Psi^{q_1+q_2}$ -module.

We omit the proof of these properties following from explicit Čech description of the above structures in N -adapted from (dubbing the constructions from [35] for h - and v -configurations). Here we note that the elements of cohomological classes, like $[e] \in H^3(\mathbf{V}, \mathbb{Z}_k) \simeq H^2(\mathbf{V}, U(1))$ and $\delta[e] \in H^3(\mathbf{V}, \mathbb{Z}_k)$, are defined for N -anholonomic manifolds, see Ref. [36] for an introduction in K -theory and related cohomological calculus. This results in distinguished (by N -connection structure) K -group of the semi-group of N -anholonomic Ψ -modules. ³

4.2 Curvatures for N -anholonomic bundle gerbe modules

For a holonomic manifold, because \mathbb{Z}_k is finite, there is a natural \check{G} -equivariant flat connection ${}^{flat}\nabla_{\mathbf{X}}$ on any cart from a covering of bundle $\mathbf{v}_{\mathbf{U}}$. For N -anholonomic manifolds the role of flat connection is played by metric canonical d_N -connection $\widehat{\mathbf{D}}^N$ defined by a (pseudo) Euclidean N -metric structure $\eta = \eta \oplus_N \bullet\eta$ and the N -connection \mathbf{N} , see Corollary 2.1. If a d -metric structure $\mathbf{g} = [g, \bullet g]$ is stated on such a N -anholonomic manifold, we shall work with the corresponding canonical d -connection $\widehat{\mathbf{D}}$, see Theorem 2.4. For simplicity, in this section we shall derive our constructions starting from $\widehat{\mathbf{D}}^N$ but we note that, in general, we can work with an arbitrary d -connection \mathbf{D} lifted on \mathbf{W} as a distinguished linear operator, N -adapted to (2), acting in the space of d -forms $\omega = (\omega^0, \omega^1, \dots)$, where, for instance, ω^1 denotes the space of 1-forms distinguished by the N -connection structure. Let us consider the d -operator

$$\overleftarrow{\mathbf{D}} : \omega^0(\mathbf{B}, \mathbf{W}) \rightarrow \omega^1(\mathbf{B}, \mathbf{W}).$$

For a necessary small open subset $\mathbf{U} \subset \mathbf{V}$, we can identify the restriction of \mathbf{B} to \mathbf{U} with $\mathbf{U} \times \mathbf{V}$ and their restriction of \mathbf{W} with $\mathbf{W}_{\mathbf{V}}$. In result, we may

³We emphasize that we wrote Ψ -modules instead of Γ -modules [1, 2, 4, 3, 35] because in this work the symbol Γ is used for d -connections.

write

$$\overleftarrow{\mathbf{D}} = \widehat{\mathbf{D}}_{\mathbf{V}}^N + \mathbf{D}_B, \quad (14)$$

for \mathbf{D}_B being a pull-back of a connection from the base \mathbf{U} . This way, $\overleftarrow{\mathbf{D}}$ is defined as a Ψ -module d-connection if it is equivariant for the group $\check{\mathbf{G}}^\# = (U(1) \times \check{\mathbf{G}}) / \mathbb{Z}_k$ with $\mathbb{Z}_k \subset U(1) \times \check{\mathbf{G}}$ parametrized as a h- and v-distinguished inclusions by anti-diagonal subgroups. Such a d-connection satisfies the rule

$$\overleftarrow{\mathbf{D}}(f\varphi) = \mathbf{e}_\mu(f)\mathbf{e}^\mu \otimes_N \varphi + f \otimes_N \overleftarrow{\mathbf{D}}\varphi \quad (15)$$

for any function f on \mathbf{V} and section φ of \mathbf{W} where \mathbf{e}_μ and \mathbf{e}^μ are N-elongated operators (23) and (24).

Let us consider two Ψ^q -module d-connections $\overleftarrow{\mathbf{D}}_1$ and $\overleftarrow{\mathbf{D}}_2$ on \mathbf{W} . The distortion $\overleftarrow{\mathbf{P}} = \overleftarrow{\mathbf{D}}_1 - \overleftarrow{\mathbf{D}}_2$ is $\check{\mathbf{G}}$ -equivariant and belongs to the d-vector space $\omega^1(\mathbf{B}, \mathbf{End}(\mathbf{W}))$, this follows from (15). For any vertical to \mathbf{V} d-vector λ , one holds $\lambda \rfloor \overleftarrow{\mathbf{P}} = 0$. This allows to "divide" on \mathbf{G} and transform $\overleftarrow{\mathbf{P}}$ into an element of $\omega^1(\mathbf{B}, \mathcal{E}(\mathbf{W}))$, i.e. by such N-adapted distortions we are able to generate all Ψ^q -module d-connections starting from (14). In result, we proved

Proposition 4.2 *The set of Ψ^q -module d-connections on \mathbf{W} is a N-distinguished affine space generated by N-adapted distortions as elements of $\omega^1(\mathbf{B}, \mathcal{E}(\mathbf{W}))$.*

The curvature of a d-connection $\overleftarrow{\mathbf{D}}$ (14) is to be constructed by globalizing the results of Theorem 2.2 (which is a very similar to the proof of the previous Proposition):

Theorem 4.1 *The curvature of a Ψ^q -module d-connections on \mathbf{W} descends to define an element $\overleftarrow{\mathbf{R}} \in \omega^2(\mathbf{B}, \mathcal{E}(\mathbf{W}))$.*

The d-connection (14) is defined by a N-adapted tensor product. This extends to a straightforward proof of a corresponding result for curvature:

Corollary 4.1 *For any Ψ^q -module d-connections $\overleftarrow{\mathbf{D}}$ and $\overleftarrow{\mathbf{D}}'$, respectively, on N-anholonomic $\check{\mathbf{G}}$ -equivariant bundles \mathbf{W} and \mathbf{W}' , we can compute the curvature of the d-tensor product connection $\overleftarrow{\mathbf{D}}$ on $\mathbf{W} \otimes \mathbf{W}'$,*

$$\mathbf{R}_B = \overleftarrow{\mathbf{R}} \otimes 1 + 1 \otimes \overleftarrow{\mathbf{R}}' \in \omega^2(\mathbf{V}, \mathcal{E}(\mathbf{W} \otimes \mathbf{W}')) = \omega^2(\mathbf{V}, \mathcal{E}(\mathbf{W}) \otimes \mathcal{E}(\mathbf{W}')).$$

In order to define the (twisted) Chern character it is enough to have the data for a Ψ -module d -connection $\overleftarrow{\mathbf{D}}$ and its descendent curvature $\overleftarrow{\mathbf{R}}$ [35, 4]. For N -anholonomic configurations, the constructions depend on the fact if there N -anholonomic manifold is provided or not with a d -metric structure.

Definition 4.2 *The (twisted and nonholonomic) Chern character of a Ψ^a -module is defined by the curvature of d -connection $\overleftarrow{\mathbf{D}}$ induced by the N -connection structure,*

$$ch(\overleftarrow{\mathbf{R}}) = tr \exp \frac{\overleftarrow{\mathbf{R}}}{2\pi i}. \quad (16)$$

Remark 4.1 *If additionally to the N -connection structure on \mathbf{V} it is defined a d -metric structure \mathbf{g} , the corresponding Chern character must be computed by using the $\overleftarrow{\mathbf{R}}'$ defined as a distortion from the nonholonomic configuration stated by a d -metric $\eta = [\eta, \bullet\eta]$ (inducing together with N the canonical d -connection $\widehat{\mathbf{D}}^N$ and $\overleftarrow{\mathbf{R}}$) to a d -connection $\mathbf{g} = [g, \bullet g]$ (inducing the canonical d -connection $\widehat{\mathbf{D}}$ and curvature $\overleftarrow{\mathbf{R}}'$).*

The values $ch(\overleftarrow{\mathbf{R}})$ and/or $ch(\overleftarrow{\mathbf{R}}')$ are closed and this mean that the corresponding de Rham cohomology classes are independent of the choice of Ψ^a -module d -connections if a N -connection structure is prescribed. This has a number of interesting applications in modern geometric mechanics, generalized Finsler geometry and gravity with nontrivial N -anholonomic structures:

Result 4.1 *Any regular Lagrange, or Finsler, configuration is topologically characterized by the corresponding canonical (twisted) Chern character (16) computed by using the curvature ${}^L\overleftarrow{\mathbf{R}}$, or ${}^F\overleftarrow{\mathbf{R}}$, induced by the curvature (7) defined by the N -connection ${}^L\mathbf{N}$, or ${}^F\mathbf{N}$, in the form (43) and d -metric ${}^L\mathbf{g}$, or ${}^F\mathbf{g}$, in the form (45) defining the canonical d -connection ${}^L\widehat{\mathbf{D}}$ (${}^F\widehat{\mathbf{D}}$).*

The set of exact solutions with generic off-diagonal metrics, nonholonomic frames and various type of local anisotropy, noncommutative and/or Lie algebroid symmetries constructed in Refs. [9, 10, 13, 14, 16, 20, 26, 11, 24] can be globalized for gerbe configurations with nontrivial N -connection structure, i.e. one holds

Result 4.2 *The geometric objects for a N -anholonomic Riemann–Cartan manifold ${}^{RC}\mathbf{V}$ can be globalized to N -anholonomic gerbe configurations and characterized by the corresponding (twisted–anholonomic) Chern character (16) computed by using the curvature $\overleftarrow{\mathbf{R}}$ induced by the curvature (7) defined by the N -connection \mathbf{N} and d -metric \mathbf{g} (8) defining the canonical d -connection $\widehat{\mathbf{D}}$.*

Finally, in this section, we conclude that the last two Results state new types of (topological) symmetries and a new classification of regular Lagrange systems, Finsler spaces and Einstein–Cartan spaces provided with N–connection structure.

5 Nonholonomic Clifford Gerbes and Modules

This section presents a development of the geometry of N–anholonomic manifolds and related nonholonomic Clifford and Dirac structures [22, 23, 14]. The reader may consult Refs. [24, 25, 26] for local component representations of the results and related local calculus and proofs.

5.1 Clifford d–algebras and N–anholonomic bundles

This work states an explicit example of generalized spinor constructions by considering in sequence (13) the d–groups $\mathbf{G} = \mathbf{Spin}(n+m)$ and $\check{\mathbf{G}} = \mathbf{SO}(n+m)$ where the boldfaced d–groups split respectively into h– and v–components $\mathbf{Spin}(n+m) = \{\mathbf{Spin}(\mathbf{n}), \mathbf{Spin}(\mathbf{m})\}$ and $\mathbf{SO}(n+m) = \{SO(n), SO(m)\}$. One get the central extension

$$0 \rightarrow \mathbb{Z}_k \rightarrow \mathbf{Spin}(n+m) \rightarrow \mathbf{SO}(\mathbf{n}+\mathbf{m}) \rightarrow 1$$

splitting into respective h– and v–components,

$$0 \rightarrow \mathbb{Z}_k \rightarrow Spin(n) \rightarrow SO(n) \rightarrow 1$$

and

$$0 \rightarrow \mathbb{Z}_k \rightarrow Spin(m) \rightarrow SO(m) \rightarrow 1.$$

Let us consider two real vector spaces v and $\bullet v$ of dimension n and m each provided with positive defined scalar products and defining a d–vector space $\mathbf{v} = v \oplus \bullet v$. We denote by $C(v)$ and $C(\bullet v)$ the corresponding \mathbb{Z}_2 graded Clifford algebras defining a Clifford d–algebra

$$\mathbf{C}(\mathbf{v}) = C_+(v) \oplus C_-(v) \oplus_N C_+(\bullet v) \oplus C_-(\bullet v).$$

The splitting \pm is related to the chirality operator $\gamma = \pm$ on C_{\pm} . A hermitian Clifford d–module is a \mathbb{Z}_2 –graded d–vector space \mathbf{v}^E provided with complex scalar products on the h– and v–components. The endomorphisms of spin representation $S = S^+ \oplus S^-$ and $\bullet S = \bullet S^+ \oplus \bullet S^-$ define respectively the hermitian Clifford modules for conventional h– and v–subspaces, $C(v) =$

$End(S)$ and $C(\bullet v) = End(\bullet S)$. Any hermitian Clifford d -modules of finite dimension can be represented in the form $\mathbf{v}^E = S \otimes \mathbf{v}^C$ where \mathbf{v}^C is a complex d -vector space on which $\mathbf{C}(\mathbf{v})$ acts trivially in distinguished from. We can identify

$$\mathbf{v}^C = Hom(\mathbf{S}, \mathbf{v}^E) \text{ and } End(\mathbf{v}^C) = End(\mathbf{v}^E)$$

supposing that such maps are h - and v -split and commute with the action of $\mathbf{C}(\mathbf{v})$.

We take the bundle \mathbf{B} from section 4.1.2 to be the bundle of N -adapted orthogonal frames (see Proposition 2.1) on $T\mathbf{V}$. In the spin case, the construction of $\check{\mathbf{G}}^\#$ is that for the d -group $\mathbf{Spin}_c(n+m)$ considered in [22, 23, 14, 25] for spin N -anholonomic manifolds, when Ψ is a trivial \mathbb{Z}_2 N -anholonomic bundle gerbe and $spin$ - c when Ψ_c is a trivial $U(1)$ bundle gerbe provided with N -connection structure.

Definition 5.1 *The lifting bundle gerbe Ψ for the case $\mathbf{G} = \mathbf{Spin}(n+m)$ and $\check{\mathbf{G}} = \mathbf{SO}(n+m)$ is called the spin-bundle N -anholonomic gerbe.*

We can consider half-spin representations S^\pm of $Spin(n)$ and $\bullet S^\pm$ of $Spin(m)$ and introduce the d -spin representations

$$\mathbf{S} = (S^+ \oplus S^-) \oplus_N (\bullet S^+ \oplus \bullet S^-). \quad (17)$$

Definition 5.2 *The Ψ^1 -modules associated to the N -adapted d -spin representation (17) define the N -anholonomic spin Ψ^1 -modules generalizing the concept of d -spin bundles on \mathbf{V} .*

The above mentioned spin constructions have a straightforward extension to even-dimensional oriented N -anholomic Riemann-Cartan manifolds (this holds always for oriented Lagrange-Finsler spaces), denoted \mathbf{V}^{2n} . One introduces the N -anholonomic bundle of complex Clifford d -algebras [14] of $T^*\mathbf{V}^{2n}$ and consider the Clifford distinguished map (multiplication) $\mathbf{c} : T^*\mathbf{V}^{2n} \rightarrow \mathbf{C}(\mathbf{V}^{2n})$, where formally $\mathbf{v} \rightarrow \mathbf{V}^{2n}$.

Definition 5.3 *A N -anholonomic Clifford module (in brief, Clifford d -module) is a complex \mathbb{Z}_2 -graded hermitian N -anholonomic vector bundle*

$$\mathbf{E} = E_+ \oplus E_- \oplus_N \bullet E_+ \oplus \bullet E_-$$

over \mathbf{V}^{2n} satisfying the properties that \mathbf{E}_u is a hermitian Clifford d -module for $\mathbf{C}_u(\mathbf{V}^{2n})$ in each point $\mathbf{u} \in \mathbf{V}^{2n}$ and that the sub-bundles E_+ and $\bullet E_+$ are respectively orthogonal to E_- and $\bullet E_-$.

We consider the spin–bundle N –anholonomic gerbe Ψ from Definition 5.1 and the pull–back of \mathbf{E} to \mathbf{B} , denoted $\mathbf{E}_{\mathbf{B}} = \pi^{-1}(\mathbf{E})$, where $\pi : \mathbf{B} \rightarrow \mathbf{V}^{2n}$ is the bundle of N –adapted frames on \mathbf{V}^{2n} . In any point $b \in \mathbf{B}$, there is an isomorphism transforming $(\mathbf{E}_{\mathbf{B}})_p$ into $\mathbf{C}(\mathbb{R}^{n+m})$ Clifford d –module. We have $(\mathbf{E}_{\mathbf{B}})_p = \mathbf{S} \otimes \mathbf{v}_{(b)}^C$ for $\mathbf{v}_{(b)}^C = Hom(\mathbf{S}, (\mathbf{E}_{\mathbf{B}})_p)$ with the homomorphisms defined on $\mathbf{C}(\mathbb{R}^{n+m})$. The construction can be globalized, $\mathbf{E}_{\mathbf{B}} = \mathbf{S} \otimes \mathbf{v}^C$. The action of $\mathbf{Spin}(n+m)$ on \mathbf{S} induce also an action $\mathbf{E}_{\mathbf{B}}$ and transforms it into a N –anholonomic Ψ^{-1} –module. In result, we proved

Theorem 5.1 *For a N –anholonomic spin bundle gerbe $\mathbf{S}_{\mathbf{B}}$, every Clifford d –module \mathbf{E} on N –anholonomic \mathbf{V}^{2n} , with its nonholonomic bundle gerbe Ψ , has the form $\mathbf{E}_{\mathbf{B}} = \mathbf{S} \otimes \mathbf{v}^C$ for some N –anholonomic Ψ^{-1} –module \mathbf{v}^C .*

This theorem generalizes for N –anholonomic spaces similar results given in [37, 4]. For spin N –anholonomic manifolds \mathbf{V}^{2n} considered in [22, 23, 14, 25], we have that every Clifford d –module is a d –tensor product of an N –anholonomic spin bundle with an arbitrary bundle.

5.2 N –anholonomic Dirac operators and gerbes

5.2.1 The index topological formula for holonomic Dirac operators

Let us remember the Atiyah–Singer index topological formula for the Dirac operator [38, 39]:

$$ind(D_{\Gamma}^+) \doteq \dim \ker(D_{\Gamma}^+) - \dim \text{co ker}(D_{\Gamma}^+) = \langle \widehat{A}(M)ch(W), [M] \rangle \quad (18)$$

where the compact M is an oriented even dimensional spin manifold with spin–bundles S^{\pm} and Γ is a unitary connection on the vector bundle W , see details on definitions and denotations in Ref. [37] (below, we shall give details for N –anholonomic configurations). In this formula, we use the genus of the manifold

$$\widehat{A}(M) \doteq \left| \det \frac{R}{2 \sinh(R/2)} \right|^{1/2}$$

determined by the Riemannian curvature R of the manifold M . The operator D_{Γ}^+ is the so–called coupled Dirac operator (first order differential operator) acting in the form $D_{\Gamma}^+ : C^{\infty}(M, E^+) \rightarrow C^{\infty}(M, E^-)$, for $E^{\pm} \doteq S^{\pm} \otimes W$. This Dirac operator can be introduced for non–spin manifolds even itself this object is not well defined. In a formal way, we can induce the Dirac

operator as a compatible connection on $E = E^+ \oplus E^-$ treated as a Clifford module with multiplication extended to act as the identity on W .

For non-spin manifolds, one exists an index formula for Dirac operators defined on hermitian Clifford modules E ,

$$\text{ind}(D_\Gamma^+) = \langle \widehat{A}(M) \text{ch}(E/S), [M] \rangle \quad (19)$$

which, if M is spin and $E^\pm \doteq S^\pm \otimes W$, the relative Chern character $\text{ch}(E/S)$ reduces to the Chern character of W , i.e. to $\text{ch}(W)$. We note that one may be not possible to define a canonical trivialization of M but it is supposed that one exist a canonical no were vanishing (volume) density $[M]$ which allows to perform integration. This always holds for the Riemannian manifolds. The aim of next section is to prove that formulas (18) and (19) can be correspondingly generalized for N-anholonomic manifolds provided with d-metric and d-connection structures.

5.2.2 Twisted nonholonomic Dirac operators on Clifford gerbes

Let us go to the Definition 5.1 of the spin-bundle N-anholonomic gerbe Ψ derived for an N-anholonomic manifold \mathbf{V} . The Clifford multiplication is parametrized by N-adapted maps between such Ψ -modules,

$$c : (\mathbb{R}_{\mathbf{B}}^n)^* \otimes S_{\mathbf{B}}^+ \rightarrow S_{\mathbf{B}}^- \text{ and } \bullet c : (\mathbb{R}_{\mathbf{B}}^m)^* \otimes \bullet S_{\mathbf{B}}^+ \rightarrow \bullet S_{\mathbf{B}}^-$$

where $\mathbb{R}_{\mathbf{B}}^{n+m} = \pi^* T\mathbf{V}$ is the bull-back to the N-adapted frame bundle from the tangent bundle $T\mathbf{V}$ with \mathbb{R}^n and \mathbb{R}^m being the fundamental representations, respectively, of $SO(n)$ and $SO(m)$ defining the d-group $\mathbf{SO}(n+m)$. Any d-connection on \mathbf{V} defines a canonical d-connection inducing a standard d-connection on the bundle of N-adapted frames \mathbf{B} .

Definition 5.4 *The N-anholonomic (twisted) Dirac operator is defined:*

$$\mathbb{D}^+ : C^\infty(\mathbf{B}, S_{\mathbf{B}}^+) \rightarrow C^\infty(\mathbf{B}, S_{\mathbf{B}}^-),$$

for Ψ -modules and

$$\overleftarrow{\mathbb{D}} : C^\infty(\mathbf{B}, S_{\mathbf{B}}^+ \otimes \mathbf{W}) \rightarrow C^\infty(\mathbf{B}, S_{\mathbf{B}}^- \otimes \mathbf{W}),$$

for \mathbf{W} being a Ψ^{-1} -module with induced canonical d-connection.

The introduced d-operators split into N-adapted components, $\mathbb{D}^+ = (h\mathbb{D}^+, \bullet\mathbb{D}^+)$ and $\overleftarrow{\mathbb{D}} = (h\overleftarrow{\mathbb{D}}, \bullet\overleftarrow{\mathbb{D}})$. These operators are correspondingly $Spin(n)$ - and $Spin(m)$ -invariant. The space $S_{\mathbf{B}}^\pm \otimes \mathbf{W}$ is a N-anholonomic

Ψ^0 -module descending to bundles \mathbf{E}^\pm on \mathbf{V} which transforms the Dirac d-operator to be a twisted Dirac d-operator:

$$\overleftarrow{\mathbb{D}}^+ : C^\infty(\mathbf{B}, \mathbf{E}^+) \rightarrow C^\infty(\mathbf{B}, \mathbf{E}^-). \quad (20)$$

If \mathbf{V} is a spin manifold, than the operators from Definition 5.4 descend to \mathbf{V} and with a local decomposition $\mathbf{E}_\mathbf{V} = \mathbf{S}_\mathbf{V} \otimes \mathbf{W}_\mathbf{V}$.

Let us consider an N-anholonomic Clifford module \mathbf{E} for $\mathbf{C}(\mathbf{V})$ for which there is a d-connection ${}^A\mathbf{D}$ induced by the canonical d-connection $\widehat{\mathbf{D}}$ and acting following the rule

$${}^A\mathbf{D}[\mathbf{c}(\omega)f] = \mathbf{c}(\widehat{\mathbf{D}}\omega)\lambda + \mathbf{c}(\omega) {}^A\mathbf{D}\lambda$$

for any 1-form ω on \mathbf{V} and $\lambda \in C^\infty(\mathbf{V}, \mathbf{E})$. This action preserves the global decomposition (2). We may associate to ${}^A\mathbf{D}$ a Dirac d-operator $\overleftarrow{\mathbb{D}}$ by using the sequence

$$C^\infty(\mathbf{V}, \mathbf{E}) \xrightarrow{{}^A\mathbf{D}} C^\infty(\mathbf{V}, T^*\mathbf{V} \otimes \mathbf{E}) \xrightarrow{\mathbf{c}} C^\infty(\mathbf{V}, \mathbf{E}).$$

This sequence splits also in h- and v-components. Because the Clifford multiplication by $T^*\mathbf{V}$ results in two distinguished odd parts of $C(\mathbf{V})$, we get an odd operator $\overleftarrow{\mathbb{D}}$ splitting in \pm components,

$$\overleftarrow{\mathbb{D}}^\pm : C^\infty(\mathbf{V}, \mathbf{E}^\pm) \rightarrow C^\infty(\mathbf{V}, \mathbf{E}^\mp)$$

acting in distinguished form on h- and v-components,

$$\overleftarrow{\mathbb{D}}^\pm : C^\infty(V, \mathbf{E}^\pm) \rightarrow C^\infty(V, \mathbf{E}^\mp) \text{ and } \overleftarrow{\mathbb{D}}^\pm : C^\infty(\bullet V, \mathbf{E}^\pm) \rightarrow C^\infty(\bullet V, \mathbf{E}^\mp).$$

Such operators are N-adapted and formal adjoint of each other respective h- and v-component of the standard functional L^2 (not confusing with Lagrange fundamental function considered in the previous section) defining the inner product on $C^\infty(\mathbf{V}, \mathbf{E})$.

The curvature ${}^A\mathbf{R}$ of the d-connection ${}^A\mathbf{D}$ is a 2-form with values in

$$End(\mathbf{E}) = \mathbf{C}(\mathbf{V}) \otimes End_{\mathbf{C}(\mathbf{V})}(\mathbf{E}).$$

In general, ${}^A\mathbf{R}$ does not commute with the action on $\mathbf{C}(\mathbf{V})$. For Riemannian manifolds, it was proposed to introduce the twisting curvature [37] $R_{E/S} = {}^A R - c(R)$ for any $c(R) \in C(M)$ satisfying the conditions $[{}^A R, c(\lambda)] = c(R(\lambda))$ and $[c(R), c(\lambda)] = c(R(\lambda))$ for R being the Riemannian curvature of M and any tangent vector λ . In a similar form, for N-anholonomic manifolds, we can define the twisting canonical curvature

$$\widehat{\mathbf{R}}_{E/S} = {}^A\mathbf{R} - c(\widehat{\mathbf{R}})$$

induced by (7), see formulas (39) from Appendix, computed for the canonical d -connection $\widehat{\mathbf{D}}$, see Theorem 2.4 and (37). With this curvature, we may act as with the Riemannian one following the procedure of defining generalized Dirac operators from [37].

5.2.3 Main result and concluding remarks

The material of previous section 5.2.2 consists the proof of

Theorem 5.2 (*Twisted Index formula for N -anholonomic Dirac operators*). *If \mathbf{V} is a compact N -anholonomic manifold, then the Dirac operator $\overleftarrow{\mathbb{D}}^+(20)$ satisfies the index formula*

$$\text{ind}(\overleftarrow{\mathbb{D}}^+) = \langle \widehat{A}(\mathbf{V}) \text{ch}(\mathbf{W}), [\mathbf{V}] \rangle,$$

where the genus

$$\widehat{A}(\mathbf{V}) \doteq \left| \det \frac{\widehat{\mathbf{R}}}{2 \sinh(\widehat{\mathbf{R}}/2)} \right|^{1/2}$$

is determined by the curvature $\widehat{\mathbf{R}}$ of the canonical d -connection $\widehat{\mathbf{D}}$.

This theorem can be stated for certain particular cases of Lagrange, or Finsler, geometries and their spinor formulation, for instance, with the aim to locally anisotropic generalization of the so-called C -spaces [40, 41] which will present topological characteristics derived from a fundamental Lagrange, or Finsler, function or, in a new fashion, for non-spin C -gerbes. The Main Result of this work can be also applied for topological classification of new classes of globalized exact solutions defining nonholonomic gravitational and matter field configurations [9, 10, 13, 14, 16, 20, 26, 11, 24] .

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A Local Formulation of N -Connection Geometry

In this Appendix, we present some important local formulas and equations defining the local geometry of N -anholonomic spaces, see details in Refs. [11, 14, 20, 28].

Locally, a N-connection, see Definition 2.1, is stated by its coefficients $N_i^a(u)$,

$$\mathbf{N} = N_i^a(u) dx^i \otimes \partial_a \quad (21)$$

where the local coordinates (in general, abstract ones both for holonomic and nonholonomic variables) are split in the form $u = (x, y)$, or $u^\alpha = (x^i, y^a)$, where $i, j, k, \dots = 1, 2, \dots, n$ and $a, b, c, \dots = n+1, n+2, \dots, n+m$ when $\partial_i = \partial/\partial x^i$ and $\partial_a = \partial/\partial y^a$. The well known class of linear connections consists on a particular subclass with the coefficients being linear on y^a , i.e., $N_i^a(u) = \Gamma_{bj}^a(x) y^b$.

An explicit local calculus allows to write the N-connection curvature (3) in the form

$$\Omega = \frac{1}{2} \Omega_{ij}^a dx^i \wedge dx^j \otimes \partial_a,$$

with the N-connection curvature coefficients

$$\Omega_{ij}^a = \delta_{[j} N_{i]}^a = \delta_j N_i^a - \delta_i N_j^a = \partial_j N_i^a - \partial_i N_j^a + N_i^b \partial_b N_j^a - N_j^b \partial_b N_i^a. \quad (22)$$

Any N-connection $\mathbf{N} = N_i^a(u)$ induces a N-adapted frame (vielbein) structure

$$\mathbf{e}_\nu = (e_i = \partial_i - N_i^a(u) \partial_a, e_a = \partial_a), \quad (23)$$

and the dual frame (coframe) structure

$$\mathbf{e}^\mu = (e^i = dx^i, e^a = dy^a + N_i^a(u) dx^i). \quad (24)$$

The vielbeins (24) satisfy the nonholonomy (equivalently, anholonomy) relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma \quad (25)$$

with (antisymmetric) nontrivial anholonomy coefficients $W_{ia}^b = \partial_a N_i^b$ and $W_{ji}^a = \Omega_{ij}^a$.⁴ These formulas present a local proof of Proposition 2.1 when

$$\mathbf{e} = \{\mathbf{e}_\nu\} = (e = \{e_i\}, \bullet e = \{e_a\})$$

and

$$\tilde{\mathbf{e}} = \{\mathbf{e}^\mu\} = (\tilde{e} = \{e^i\}, \bullet \tilde{e} = \{e^a\}).$$

Let us consider metric structure

$$\check{g} = \underline{g}_{\alpha\beta}(u) du^\alpha \otimes du^\beta \quad (26)$$

⁴One preserves a relation to our previous denotations [22, 23] if we consider that $\mathbf{e}_\nu = (e_i, e_a)$ and $\mathbf{e}^\mu = (e^i, e^a)$ are, respectively, the former $\delta_\nu = \delta/\partial u^\nu = (\delta_i, \partial_a)$ and $\delta^\mu = \delta u^\mu = (d^i, \delta^a)$ when emphasize that operators (23) and (24) define, correspondingly, the ‘‘N-elongated’’ partial derivatives and differentials which are convenient for calculations on N-anholonomic manifolds.

defined with respect to a local coordinate basis $du^\alpha = (dx^i, dy^a)$ by coefficients

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}. \quad (27)$$

In general, such a metric (27) is generic off-diagonal, i.e. it can not be diagonalized by any coordinate transforms and that $N_i^a(u)$ are any general functions. The condition (9), for $X \rightarrow e_i$ and $\bullet Y \rightarrow \bullet e_a$, transform into

$$\check{g}(e_i, \bullet e_a) = 0, \text{ equivalently } \underline{g}_{ia} - N_i^b h_{ab} = 0$$

where $\underline{g}_{ia} \doteq g(\partial/\partial x^i, \partial/\partial y^a)$, which allows to define in a unique form the coefficients $N_i^b = h^{ab} \underline{g}_{ia}$ where h^{ab} is inverse to h_{ab} . We can write the metric \check{g} with ansatz (27) in equivalent form, as a d-metric adapted to a N-connection structure, see Definition 2.7,

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}(u) \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{ij}(u) e^i \otimes e^j + h_{ab}(u) \bullet e^a \otimes \bullet e^b, \quad (28)$$

where $g_{ij} \doteq \mathbf{g}(e_i, e_j)$ and $h_{ab} \doteq \mathbf{g}(\bullet e_a, \bullet e_b)$ and the vielbeins \mathbf{e}_α and \mathbf{e}^α are respectively of type (23) and (24).

We can say that the metric \check{g} (26) is equivalently transformed into (28) by performing a frame (vielbein) transform

$$\mathbf{e}_\alpha = \mathbf{e}_\alpha^\alpha \underline{\partial}_\alpha \text{ and } \mathbf{e}^\beta = \mathbf{e}^\beta_\beta du^\beta.$$

with coefficients

$$\mathbf{e}_\alpha^\alpha(u) = \begin{bmatrix} e_i^i(u) & N_i^b(u) e_b^a(u) \\ 0 & e_a^a(u) \end{bmatrix}, \quad (29)$$

$$\mathbf{e}^\beta_\beta(u) = \begin{bmatrix} e^i_i(u) & -N_k^b(u) e^k_i(u) \\ 0 & e^a_a(u) \end{bmatrix}, \quad (30)$$

being linear on N_i^a . We can consider that a N-anholonomic manifold \mathbf{V} provided with metric structure \check{g} (26) (equivalently, with d-metric (28)) is a special type of a manifold provided with a global splitting into conventional ‘‘horizontal’’ and ‘‘vertical’’ subspaces (2) induced by the ‘‘off-diagonal’’ terms $N_i^b(u)$ and a prescribed type of nonholonomic frame structure (25).

A d-connection, see Definition 2.3, splits into h- and v-covariant derivatives, $\mathbf{D} = D + \bullet D$, where $D_k = (L_{jk}^i, L_{bk}^a)$ and $\bullet D_c = (C_{jk}^i, C_{bc}^a)$ are correspondingly introduced as h- and v-parametrizations of (31),

$$L_{jk}^i = (\mathbf{D}_k e_j)] e^i, \quad L_{bk}^a = (\mathbf{D}_k e_b)] e^a, \quad C_{jc}^i = (\mathbf{D}_c e_j)] e^i, \quad C_{bc}^a = (\mathbf{D}_c e_b)] e^a.$$

The components $\Gamma_{\alpha\beta}^\gamma = (L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a)$ completely define a d-connection \mathbf{D} on a N-anholonomic manifold \mathbf{V} .

The simplest way to perform a local covariant calculus by applying d-connections is to use N-adapted differential forms like $\Gamma_\beta^\alpha = \Gamma_{\beta\gamma}^\alpha \mathbf{e}^\gamma$ with the coefficients defined with respect to (24) and (23). One can introduce the d-connection 1-form

$$\Gamma_\beta^\alpha = \Gamma_{\beta\gamma}^\alpha \mathbf{e}^\gamma,$$

when the N-adapted components of d-connection $\mathbf{D}_\alpha = (\mathbf{e}_\alpha \rfloor \mathbf{D})$ are computed following formulas

$$\Gamma_{\alpha\beta}^\gamma(u) = (\mathbf{D}_\alpha \mathbf{e}_\beta) \rfloor \mathbf{e}^\gamma, \quad (31)$$

where " \rfloor " denotes the interior product. This allows to define in local form the torsion (5) $\mathbf{T} = \{\mathcal{T}^\alpha\}$, where

$$\mathcal{T}^\alpha \doteq \mathbf{D}\mathbf{e}^\alpha = d\mathbf{e}^\alpha + \Gamma_\beta^\alpha \wedge \mathbf{e}^\beta \quad (32)$$

and curvature (7) $\mathbf{R} = \{\mathcal{R}^\alpha_\beta\}$, where

$$\mathcal{R}^\alpha_\beta \doteq \mathbf{D}\Gamma_\beta^\alpha = d\Gamma_\beta^\alpha - \Gamma_\beta^\gamma \wedge \Gamma_\gamma^\alpha. \quad (33)$$

The d-torsions components of a d-connection \mathbf{D} , see Theorem 2.1, are computed

$$\begin{aligned} T_{jk}^i &= L_{jk}^i - L_{kj}^i, \quad T_{ja}^i = -T_{aj}^i = C_{ja}^i, \quad T_{ji}^a = \Omega_{ji}^a, \\ T_{bi}^a &= T_{ib}^a = \frac{\partial N_i^a}{\partial y^b} - L_{bi}^a, \quad T_{bc}^a = C_{bc}^a - C_{cb}^a. \end{aligned} \quad (34)$$

For instance, T_{jk}^i and T_{bc}^a are respectively the coefficients of the $h(hh)$ -torsion $T(X, Y)$ and $v(vv)$ -torsion $\bullet T(\bullet X, \bullet Y)$.

The Levi-Civita linear connection $\nabla = \{\nabla \Gamma_{\beta\gamma}^\alpha\}$, with vanishing both torsion and nonmetricity $\nabla \check{g} = 0$, is not adapted to the global splitting (2). There is a preferred, canonical d-connection structure, $\widehat{\mathbf{D}}$, on N-anholonomic manifold \mathbf{V} constructed only from the metric and N-connection coefficients $[g_{ij}, h_{ab}, N_i^a]$ and satisfying the conditions $\widehat{\mathbf{D}}\mathbf{g} = 0$ and $\widehat{T}_{jk}^i = 0$ and $\widehat{T}_{bc}^a = 0$, see Theorem 2.4. By straightforward calculations with respect to the N-adapted bases (24) and (23), we can verify that the connection

$$\widehat{\Gamma}_{\beta\gamma}^\alpha = \nabla \Gamma_{\beta\gamma}^\alpha + \widehat{\mathbf{P}}_{\beta\gamma}^\alpha \quad (35)$$

with the deformation d-tensor ⁵

$$\widehat{\mathbf{P}}_{\beta\gamma}^\alpha = (P_{jk}^i = 0, P_{bk}^a = e_b(N_k^a), P_{jc}^i = -\frac{1}{2}g^{ik}\Omega_{kj}^a h_{ca}, P_{bc}^a = 0) \quad (36)$$

⁵ $\widehat{\mathbf{P}}_{\beta\gamma}^\alpha$ is a tensor field of type (1,2). As is well known, the sum of a linear connection and a tensor field of type (1,2) is a new linear connection.

satisfies the conditions of the mentioned Theorem. It should be noted that, in general, the components \widehat{T}_{ja}^i , \widehat{T}_{ji}^a and \widehat{T}_{bi}^a are not zero. This is an anholonomic frame (or, equivalently, off-diagonal metric) effect. With respect to the N-adapted frames, the coefficients

$\widehat{\Gamma}_{\alpha\beta}^\gamma = (\widehat{L}_{jk}^i, \widehat{L}_{bk}^a, \widehat{C}_{jc}^i, \widehat{C}_{bc}^a)$ are computed:

$$\begin{aligned}\widehat{L}_{jk}^i &= \frac{1}{2}g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \\ \widehat{L}_{bk}^a &= e_b(N_k^a) + \frac{1}{2}h^{ac}(e_k h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d), \\ \widehat{C}_{jc}^i &= \frac{1}{2}g^{ik} e_c g_{jk}, \quad \widehat{C}_{bc}^a = \frac{1}{2}h^{ad}(e_c h_{bd} + e_c h_{cd} - e_d h_{bc}).\end{aligned}\tag{37}$$

In some approaches to Finsler geometry [32], one uses the so-called Berwald d-connection \mathbf{D}^B with the coefficients

$${}^B\mathbf{\Gamma}_{\alpha\beta}^\gamma = \left({}^B L_{jk}^i = \widehat{L}_{jk}^i, {}^B L_{bk}^a = e_b(N_k^a), {}^B C_{jc}^i = 0, {}^B C_{bc}^a = \widehat{C}_{bc}^a \right).\tag{38}$$

This d-connection minimally extends the Levi-Civita connection (it is just the Levi-Civita connection if the integrability conditions are satisfied, i.e. $\Omega_{kj}^a = 0$, see (36)). But, in general, for this d-connection the metricity conditions are not satisfied, for instance $D_a g_{ij} \neq 0$ and $D_i h_{ab} \neq 0$.

By a straightforward d-form calculus in (33), we can find the N-adapted components $\mathbf{R}_{\beta\gamma\delta}^\alpha$ of the curvature $\mathbf{R} = \{\mathcal{R}_\beta^\alpha\}$ of a d-connection \mathbf{D} , i.e. the d-curvatures from Theorem 2.2:

$$\begin{aligned}R^i{}_{hjk} &= e_k L^i{}_{hj} - e_j L^i{}_{hk} + L^m{}_{hj} L^i{}_{mk} - L^m{}_{hk} L^i{}_{mj} - C^i{}_{ha} \Omega^a{}_{kj}, \\ R^a{}_{bjk} &= e_k L^a{}_{bj} - e_j L^a{}_{bk} + L^c{}_{bj} L^a{}_{ck} - L^c{}_{bk} L^a{}_{cj} - C^a{}_{bc} \Omega^c{}_{kj}, \\ R^i{}_{jka} &= e_a L^i{}_{jk} - D_k C^i{}_{ja} + C^i{}_{jb} T^b{}_{ka}, \\ R^c{}_{bka} &= e_a L^c{}_{bk} - D_k C^c{}_{ba} + C^c{}_{bd} T^d{}_{ka}, \\ R^i{}_{jbc} &= e_c C^i{}_{jb} - e_b C^i{}_{jc} + C^h{}_{jb} C^i{}_{hc} - C^h{}_{jc} C^i{}_{hb}, \\ R^a{}_{bcd} &= e_d C^a{}_{bc} - e_c C^a{}_{bd} + C^e{}_{bc} C^a{}_{ed} - C^e{}_{bd} C^a{}_{ec}.\end{aligned}\tag{39}$$

Contracting respectively the components of (39), one proves

The Ricci tensor $\mathbf{R}_{\alpha\beta} \doteq \mathbf{R}^\tau{}_{\alpha\beta\tau}$ is characterized by h- v-components, i.e. d-tensors,

$$R_{ij} \doteq R^k{}_{ijk}, \quad R_{ia} \doteq -R^k{}_{ika}, \quad R_{ai} \doteq R^b{}_{aib}, \quad R_{ab} \doteq R^c{}_{abc}.\tag{40}$$

It should be noted that this tensor is not symmetric for arbitrary d-connections \mathbf{D} .

The scalar curvature of a d-connection is

$${}^s\mathbf{R} \doteq \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab}, \quad (41)$$

defined by a sum the h- and v-components of (40) and d-metric (28).

The Einstein tensor is defined and computed in standard form

$$\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} {}^s\mathbf{R} \quad (42)$$

For a Lagrange geometry, see Definition 3.1, by straightforward component calculations, one can be proved the fundamental results:

1. The Euler–Lagrange equations

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0$$

where $y^i = \frac{dx^i}{d\tau}$ for $x^i(\tau)$ depending on parameter τ , are equivalent to the “nonlinear” geodesic equations

$$\frac{d^2 x^i}{d\tau^2} + 2G^i(x^k, \frac{dx^j}{d\tau}) = 0$$

defining paths of the canonical semispray

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$$

where

$$2G^i(x, y) = \frac{1}{2} {}^L g^{ij} \left(\frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right)$$

with ${}^L g^{ij}$ being inverse to (11).

2. There exists on \widetilde{TM} a canonical N-connection

$${}^L N_j^i = \frac{\partial G^i(x, y)}{\partial y^j} \quad (43)$$

defined by the fundamental Lagrange function $L(x, y)$, which prescribes nonholonomic frame structures of type (23) and (24), ${}^L \mathbf{e}_\nu = (e_i, \bullet e_k)$ and ${}^L \mathbf{e}^\mu = (e^i, \bullet e^k)$.

3. The canonical N-connection (43), defining $\bullet e_i$, induces naturally an almost complex structure $\mathbf{F} : \chi(\widetilde{TM}) \rightarrow \chi(\widetilde{TM})$, where $\chi(\widetilde{TM})$ denotes the module of vector fields on \widetilde{TM} ,

$$\mathbf{F}(e_i) = \bullet e_i \text{ and } \mathbf{F}(\bullet e_i) = -e_i,$$

when

$$\mathbf{F} = \bullet e_i \otimes e^i - e_i \otimes \bullet e^i \quad (44)$$

satisfies the condition $\mathbf{F} \lrcorner \mathbf{F} = -\mathbf{I}$, i. e. $F^\alpha_\beta F^\beta_\gamma = -\delta^\alpha_\gamma$, where δ^α_γ is the Kronecker symbol and “ \lrcorner ” denotes the interior product.

4. On \widetilde{TM} , there is a canonical metric structure

$${}^L\mathbf{g} = {}^Lg_{ij}(x, y) e^i \otimes e^j + {}^Lg_{ij}(x, y) \bullet e^i \otimes \bullet e^j \quad (45)$$

constructed as a Sasaki type lift from M .

5. There is also a canonical d-connection structure ${}^L\widehat{\mathbf{F}}^\gamma_{\alpha\beta}$ defined only by the components of ${}^LN_j^i$ and ${}^Lg_{ij}$, i.e. by the coefficients of metric (45) which in its turn is induced by a regular Lagrangian. The values ${}^L\widehat{\mathbf{F}}^\gamma_{\alpha\beta} = ({}^L\widehat{L}_{jk}^i, {}^L\widehat{C}_{bc}^a)$ are computed just as similar values from (37). We note that on \widetilde{TM} there are couples of distinguished sets of h- and v-components.

References

- [1] M. K. Murray, J. London Math. Soc. **54** (1996) 403
- [2] P. Pouwknegt, A. L. Carey, V. Mathai, V. K. Murray and D. Stevenson, Commun. Math. Phys. **228** (2002) 17
- [3] A. Tsemo, An Atiyah–Singer Theorem for Gerbes, math.DG/ 0302050
- [4] M. Murray and M. Singer, Gerbes, Clifford Modules and the Index Theorem, math.DG/0302096
- [5] J. W. Barrett, Int. J. Theor. Phys. **30** (1991) 1171
- [6] A. Caetano and R. F. Picken, Int. J. Math. **5** (1994)
- [7] M. Mackaay and R. Picken, Adv. Math. **170** (2002) 287

- [8] L. Breen and W. Messing, Differential Geometry of Gerbes, math.AG/0106083
- [9] S. Vacaru, JHEP, **04** (2001) 009
- [10] S. Vacaru and D. Singleton, J. Math. Phys. **43** (2002) 2486
- [11] S. Vacaru, E. Gaburov and D. Gontsa, A Method of Constructing Off-Diagonal Solutions in Metric-Affine and String Gravity, hep-th/0310133
- [12] M. Kontsevich, Lett. Math. Phys. **56** (2001) 271
- [13] F. Etayo, R. Santamaría and S. Vacaru, J. Math. Phys. **46** (2005) 032901
- [14] S. Vacaru, Nonholonomic Clifford Structures and Noncommutative Riemann–Finsler Geometry, math.DG/0408121
- [15] S. Vacaru, Phys. Lett. **B 498** (2001) 74
- [16] S. Vacaru, J. Math. Phys. **46** (2005) 042503
- [17] S. Vacaru, Nucl. Phys. B, **434** (1997) 590
- [18] A. Bejancu, Finsler Geometry and Applications (Ellis Harwood Ltd., 1990)
- [19] E. Esrafilian and E. Azizpour, Rep. Math. Phys. **54** (2004) 365
- [20] S. Vacaru, Nonholonomic Deformations of Disk Solutions and Algebroid Symmetries in Einstein and Extra Dimension Gravity, gr-qc/0504095
- [21] A. Carey, J. Mickelsson and M. Murray, Rev. Math. Phys. **12** (2000) 65
- [22] S. Vacaru, J. Math. Phys. **37** (1996) 508
- [23] S. Vacaru, JHEP, **09** (1998) 011
- [24] S. Vacaru and P. Stavrinou, Spinors and Space–Time Anisotropy (Athens University Press, Athens, Greece, 2002), 301 pages, gr-qc/0112028
- [25] S. Vacaru and N. Vicol, Int. J. Math. and Math. Sciences **23** (2004) 1189; math.DG/0406585
- [26] S. Vacaru, Clifford Algebroids and Nonholonomic Einstein–Dirac Structures, hep-th/0501217
- [27] E. Cartan, Les Espaces de Finsler (Paris, Hermann, 1935)

- [28] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications* (Kluwer, 1994)
- [29] C. Ehresmann, *Les connexions infinitesimales dans un espace fibre differentiable*. Coll. Topologia, Bruxelles (1955), p. 29
- [30] M. de Leon, M. McLean, L. K. Norris, A. Rey-Roca and M. Salgado, *Geometric Structures in Field Theory*, math-ph/020803
- [31] J. Grifone, *Ann. Inst. Fourier, Grenoble*, **22** (1972) 291
- [32] D. Bao, S. S. Chern and Z. Chen, *An Introduction to Riemann–Finsler Geometry* (Springer, 2000)
- [33] J. Kern, *Arch. Math.* **25** (1974) 438
- [34] C. Barcelo, S. Liberaty and M. Visser, *Analogie Gravity*, gr-qc/0505065
- [35] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray and D. Stevenson, *Com. Math. Phys.* **228** (2002) 17
- [36] M. Karoubi, *K–Theory: An Introduction* (Springer, 1978)
- [37] N. Berline, E. Getzler and M. Vergé, *Heat Kernels and Dirac Operators* (Springer–Verlag, 1992)
- [38] M. F. Atiyah and I. M. Singer, *Bull. Amer. Math. Soc.* **69** (1963) 422
- [39] M. F. Atiyah and I. M. Singer, *Ann. Math.* **87** (1968) 484
- [40] C. Castro, *Chaos Solitons Fractals* 11 (2000) 1663, hep-th/0001134
- [41] C. Castro and M. Pavšič, *Phys. Lett. B* 359 (2002) 133