RUNNING NEWTONIAN COUPLING AND HORIZONLESS SOLUTIONS IN QUANTUM EINSTEIN GRAVITY

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Abstract

It is shown how the exact Nonperturbative Renormalization Group flow of the running Newtonian coupling $G(r)$ in Quantum Einstein Gravity is consistent with the existence of an ultra-violet cutoff $R(r=0) = 2G_N M_o$ in the most general Schwarzschild solutions. After setting $g_{tt} = 1 - 2G_N M_o/R(r) = 1 - 2G(r)M(r)/r$, and due to the condition $G(r = 0) = 0$ and $M(r=0) \sim 1/2G_N M_o$, we prove why there is *no* horizon, since $g_{tt}(r=0) = 0$, and there is a delta function scalar curvature singularity at $r = 0$. Similar results follow in generalized Anti de Sitter-Schwarzschild metrics with a running cosmological parameter $\Lambda(r)$ and Newtonian coupling $G(r)$. The ultra-violet cutoff in this latter case is no longer given by $2G_N M_o$, but instead is given by a real-valued positive root R_* of a cubic equation associated with the condition $g_{tt}(R(r=0)) = g_{tt}(R_*) = 0$. A running Newtonian coupling $G(r)$ can also be accommodated naturally in a Jordan-Brans-Dicke scalar-tensor theory of Gravity via a trivial conformal transformation of the Schwarzschild metric. However, the running Newtonian coupling $G(r) = (16\pi\Phi^2)^{-1}$ corresponding to the scalar field Φ does not satisfy the asymptotic freedom condition $G(r=0) = 0$ associated with the ultra-violet non-Gaussian fixed point of Nonperturbative Quantum Einstein Gravity. Nevertheless, our results exhibit an interesting ultra-violet/infrared *duality* behaviour of $G(r)$ that warrants further investigation. Some final remarks are added pertaining naked singularities in higher derivative gravity, Finsler geometry, metrics in phase spaces and the connection between an ultra-violet cutoff in Noncommutative spacetimes and the general Schwarzschild solutions.

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1 Renormalization Group Flow and Schwarzschild solution

1.1 Introduction

We begin by writing down the class of static spherically symmetric (SSS) solutions of Einstein's equations [1] studied by [5], [8], [7], [6] among others, and most recently [12] given by a *infinite* family of solutions parametrized by a family of admissible radial functions $R(r)$

$$(ds)^2 = g_{00}\ (dt)^2\ - g_{RR}\ (dR)^2\ - R^2\ (d\Omega)^2 =$$

$$g_{00}\ (dt)^2\ - g_{RR}\ (\frac{dR}{dr})^2\ (dr)^2\ - R^2\ (d\Omega)^2 = g_{00}\ (dt)^2\ - g_{rr}\ (dr)^2\ - (R(r))^2\ (d\Omega)^2 \quad (1.1a)$$

where the solid angle infinitesimal element is

$$(d\Omega)^2 = sin^2(\phi)(d\theta)^2 + (d\phi)^2. \quad (1.1b)$$

and

$$g_{00} = (1 - \frac{2\ G_N\ M_o}{R(r)})\ ; \quad g_{RR} = \frac{1}{g_{00}} = \frac{1}{1-(2\ G_N\ M_o/R(r))}.$$

$$g_{rr} = g_{RR}\ (dR/dr)^2 = (1 - \frac{2\ G_N\ M_o}{R(r)})^{-1}\ (\frac{dR(r)}{dr})^2. \quad (1.1c)$$

Notice that the static spherically symmetric (SSS) vacuum solutions of Einstein's equations, with and without a cosmological constant, do *not* determine the form of the radial function $R(r)$ [12], [10]. There are two classes of solutions; (i) those solutions whose radial functions obey the condition $R(r=0)=0$, like the Hilbert textbook black hole solution $R(r)=r$ with a horizon at $r=2G_NM_o$; and (ii) those horizonless solutions with an ultraviolet cutoff $R(r=0)=2G_NM_o$. In particular, for radial functions like

$$R(r) = r + 2G_NM_o; \quad R(r) = [r^3 + (2G_NM_o)^3]^{1/3}; \quad R(r) = \frac{2G_NM_o}{1-e^{-2G_NM_o/r}}. \quad (1.2)$$

found by Brillouin [3] , Schwarzschild [2] and Fiziev-Manev [7] respectively obeying the conditions that $R(r=0)=2G_NM_o$ and when $r >> 2G_NM_o \Rightarrow R(r) \rightarrow r$.

It is very important to emphasize that despite the fact that one can always *relabel* the variable r for R in such a way that the metric in eq-(1.1) has exactly the *same functional form* as the standard Hilbert textbook solution [4] (black-holes solutions with a horizon at $r=2G_NM_o$) this does *not* mean that the Hilbert textbook metric is *diffeomorphic* to the metric in eq-(1.1). The reason is that the values of r range from 0 to ∞ while the values of R range from $2G_NM_o$ to ∞. The physical explanation why there is an ultra-violet cutoff at $R=2G_NM_o$ was provided long ago by Abrams [5], and rather than imposing this cutoff $R=2G_NM_o$ by fiat (by decree, by hand) there is a deep physical reason for doing so; namely it has been argued that the Hilbert textbook solution $R(r) = r$ does not properly represent the static gravitational field of a point mass centered at the origin $r = 0$ [5], [7], [8], [6] and the Hilbert textbook solution is not *static* in the region $0 < r < 2G_N M_o$ after performing the Fronsdal-Kruskal-Szekeres analytical continuation in terms of the new u, v coordinates.

There are many *physical* differences among the Hilbert textbook solution that has a horizon at $r = 2G_N M_o$ and the original 1916 Schwarzchild's *horizonless* solution [2]. The Schwarzschild 1916 solution is *not* a naive radial reparametrization of the Hilbert solution because the radial function chosen by Schwarzschild $R^3 = |r|^3 + (2G_N M_o)^3$ can *never zero*. The absolute value $|r|$ properly accounts for the field of a point mass source located at $r = 0$. Thus, the lower bound of R is given by $2G_N M_o$, and R cannot be zero for a nonvanishing point mass source.

The Fronsdal-Kruskal-Szekeres analytical continuation of the Hilbert textbook solution for $r < 2G_N M_o$ yields a *spacelike* singularity at $r = 0$ and the roles of t and r are interchanged when one crosses $r = 2G_N M_o$; so the interior region $r < 2G_N M_o$ is *no* longer *static*. The Schwarzchild solution is *static* for all values of r and in particular for $r < 2G_N M_o$; there is *no* horizon at $r = 2G_N M_o$ and there is a *timelike* naked singularity at $r = 0$, the true location of the point mass source. Notice that when $r >> 2G_N M_o$ the Schwarzchild solution reduces to the Hilbert solution and one has the correct Newtonian limt.

Colombeau [11] developed the rigorous mathematical treatment of tensor-valued distributions in General Relativity, new generalized functions (nonlinear distributional geometry) and multiplication of distributions in *nonlinear* theories like General Relativity since the the standard Schwarz theory of linear distributions is invalid in nonlinear theories. This treatment is essential in order to understand the physical singularity at the point-mass location $r = 0$. In [10] we studied the many subtleties behind the introduction of a true point-mass source at $r = 0$ (that couples to the vacuum field) and the physical consequences of the delta function singularity (of the scalar curvature) at the location of the point mass source $r = 0$. Those solutions were obtained from the vacuum SSS solutions simply by replacing r for $|r|$. For instance, the Laplacian in spherical coordinates in flat space of $1/|r|$ is equal to $-(1/r^2)\delta(r)$, but the Laplacian of $1/r$ is *zero*. Thus, to account for the presence of a true mass-point source at $r = 0$ one must use solutions depending on the modulus $|r|$ instead of r.

One can have an infinite number of metrics parametrized by a family of arbitrary radial functions $R(r)$ with the desired behaviour at $r = 0$ and $r = \infty$, whose values for the scalar curvature (parametrized by a family of arbitrary radial functions $R(r)$) are given by [10]

$$\mathcal{R} \;=\; -\frac{2\; G_N\; M_o\; \delta(r)}{R^2\;(dR/dr)};\; in\ units\ of\ c = 1. \qquad (1.3a)$$

Since the scalar curvature $\mathcal{R}$ (1.3a) is a *coordinate invariant* quantity, this result in eq-(1.3a) that depends explicitly on the family of radial functions $R(r)$ corroborates once more that one cannot view the role of the radial function $R(r)$ as a naive change of radial coordinates from r to R. Hence, one must view the radial function squared $R^2(r)$ as just one of the *metric* tensor-field components $g_{\phi\phi}(r) \equiv R^2(r)$; i.e. $R(r)^2$ is a *function* of the radial coordinate r that has a lower cutoff given by $g_{\phi\phi}(r = 0) = (2G_N M_o)^2$. One must not confuse R with r and even after relabeling r for R, the metric in eq-(1.1) is *not* diffeomorphic to the Hilbert textbook solution due to the cutoff $R = 2G_N M_o$. If one chooses the radial functions to obey the condition $R(r = 0) = 0$ and $R(r \to \infty) \sim r$ then only in this case these metrics are diffeomorphic to the Hilbert textbook black hole solution.

The relevant *invariant* physical quantity *independent* of the any *arbitrary* choice of $R(r)$ is the Einstein-Hilbert action, whether it obeys the condition $R(r = 0) = 0$ or $R(r = 0) = 2G_N M_o$. In particular, the Euclideanized action after a compactification of the temporal interval yields an invariant quantity which is precisely equal to the "black hole" entropy in Planck area units. The invariant area is the proper area at $r = 0$ given by $4\pi R(r = 0)^2 = 4\pi(2G_N M_o)^2$. We shall see that the source of entropy is due entirely to the scalar curvature delta function singularity at the location of the point mass source given by $\mathcal{R} = -[2G_N M_o/R^2(dR/dr)]\delta(r)$ [10] after using the 4-dim measure $4\pi R^2\ (|g_{RR}|^{1/2}dR)\ (|g_{tt}|^{1/2}dt) = 4\pi R^2\ dR\ dt$ in the Euclidean Einstein-Hilbert action.

Therefore, the Einstein-Hilbert action associated with the scalar curvature delta function in eq-(1.3a) when the four-dim measure is

$$d^4x \ = \ 4\pi\ R^2\ dR\ dt. \tag{1.3b}$$

is

$$S = -\frac{1}{16\pi G_N} \int \ (4\pi\ R^2\ dR\ dt)\ (-\frac{2M\ \delta(r)}{R^2\ (dR/dr)}) \ = \\ \frac{1}{16\pi G_N} \int \ (\ \frac{2G_N M_o}{r^2}\ \delta(r)\ )\ (4\pi r^2\ dr\ dt). \tag{1.3c}$$

Notice that the action (1.3c) is truly *invariant* and *independent* of any *arbitrary* choice of the radial function $R(r)$, whether or not it is the Hilbert textbook choice $R(r) = r$, or any other choice for $R(r)$. The Euclideanized action (1.3c) becomes, after reinserting the Newtonian coupling $G = L^2_{Planck}$ in order to have the proper units,

$$S(Euclidean) \ = \ \frac{4\pi(G_N M_o)^2}{G_N} \ = \ \frac{4\pi\ (2G_N M_o)^2}{4\ L^2_{Planck}} = \frac{Area}{4\ L^2_{Planck}}. \tag{1.3d}$$

when the Euclidean time coordinate interval $2\pi t_E$ is defined in terms of the Hawking temperature T_H and Boltzman constant k_B as $2\pi t_E = (1/k_B T_H) = 8\pi G_N M_o$. It is interesting that the Euclidean action (1.3c) is the same as the "black hole" entropy (1.3d) in Planck area units. The source of entropy is due entirely to the scalar curvature delta function singularity at the location of the point mass source. Furthermore, this result that the Euclidean action is equal to the entropy in Planck units can be generalized to higher dimensions upon recurring to Schwarzschild-like metrics in higher dimensions.

The fact that a point-mass can have a non-zero *proper area* $4\pi R(r = 0)^2 = 4\pi(2G_N M_o)^2$, but no volume, due to the metric and curvature singularity at $r = 0$ seems to indicate a *stringy* nature underlying the very notion of a point-mass itself. The string world-sheet has a non-zero area but zero volume. Aspinwall [13] has studied how a string (an extended object) can probe space-time points due to the breakdown of our ordinary concepts of Topology at small scales. In [12] it was shown how the Bars-Witten *stringy* $1+1$-dim black-hole metric [14] can be embedded into the 4-dim *conformally* re-scaled metrics displayed in eq-(1.1), if and only if, the radial function $R(r)$ was given implicitly by the following relationship involving R and r (the left hand side has the same functional form as the radial tortoise coordinate) :

$$\int \frac{dR}{1 - 2G_N M_o/R} \;=\; R \;+\; 2G_N M_o \; ln \; (\frac{R - 2G_N M_o}{2M_o}) \;=\; 2G_N M_o \; ln \; [\; sinh \; \frac{r}{2G_N M_o} \;]. \tag{1.4a}$$

one can verify that there is an ultra-violet cutoff at $r = 0$

$$R(r = 0) \;=\; 2G_N M_o; \quad R(r \to \infty) \;\to\; R \;\sim\; r. \tag{1.4b}$$

which *precisely* has the same behaviour at $r = 0$ and ∞ as the radial functions displayed in this section. The fact that the stringy black-hole $1 + 1$-dim solution can be embedded into the conformally rescaled solutions of this section, for a very specific functional form of the radial function $R(r)$, with the same "boundary" conditions at $r = 0$ and $r = \infty$ as the radial functions displayed in this section, is very appealing. Similar conclusions apply to horizonless solutions in higher dimensions $D > 4$ [12] with a cutoff $R(r = 0) = [16\pi G_D M_o/(D-2)\Omega_{D-2}]^{1/D-3}$ where the point-mass has a nonzero $D-2$-dimensional measure and a zero $D-1$-dim "volume". The point-mass in this case is $p - branelike$ in nature with $p + 1 = D - 2$. For example, in $D = 5$ one has a membrane-like behaviour of a point mass. In $D = 6$ one has a 3-brane-like behaviour of a point mass, etc.... The $D = 4$ case is special since it corresponds to the string.

1.2 Renormalization Group Flow and Horizonless Solutions

The purpose of this section is to explain the meaning of the ultra-violet cutoff $R(r = 0) = 2G_N M_o$ within the context of the exact Nonperturbative Renormalization Group flow of the Newtonian coupling $G = G(r)$ in Quantum Einstein Gravity [16] where a non-Gaussian ultra-violet fixed point was found $G(r = 0) = 0$. The presence of an ultra-violet cutoff $R = 2G_N M_o$ originates from the mere presence of matter and permits to relate the metric component $g_{tt} = 1 - 2G_N M_o/R(r)$ to $g_{tt} = 1 - 2G(r)M(r)/r$, in such a way that the the small distance behaviour of $G(r)$ *eliminates* the presence of a horizon at $r = 2G_N M_o$: we will see why the metric component g_{tt} evaluated at the location of the point mass source $r = 0$ is $g_{tt}(r = 0) = 0$, due to $G(r = 0) = 0, M(r = 0) = finite$ but it does *not* eliminate the delta function singularity of the scalar curvature at $r = 0$. This result is compatible with the ultra-violet cutoff of the radial function $R(r = 0) = 2G_N M$. G_N is the value of the Newtonian coupling in the deep infrared and $M = M_o$ is the Kepler mass as seen by an observer at asymptotic infinity.

The momentum dependence of $G(k^2)$ was found by Reuter et al [16] to be

$$G(k^2) = \frac{G_N}{1 + \alpha\ G_N\ k^2}. \qquad (1.5a)$$

The momentum-scale relationship is defined

$$k^2 \ = \ (\frac{\beta}{D(R)})^2, \quad \beta = constant. \qquad (1.5b)$$

in terms of the proper radial distance $D(R)$

$$D(R) = \int_{2G_N M_o}^{R} \sqrt{g_{RR}}\ dR \ = \ \int_{2G_N M_o}^{R} \frac{dR}{\sqrt{1 - (2\ G_N\ M_o/R)}} =$$

$$\sqrt{R\ (\ R - 2\ G_N\ M_o)} \ + \ 2\ G_N\ M_o\ ln\ [\ \sqrt{\frac{R}{2G_N M_o}} \ + \sqrt{\frac{R - 2G_N M_o}{2G_N M_o}}\ ]. \qquad (1.6)$$

where the lower (ultra-violett cutoff) is $R(r = 0) = 2G_N M_o$. The proper distance corresponding to $r = 0$ is $D(R(r = 0)) = D(R = 2G_N M_o) = 0$ as it should since the proper distance from $r = 0$ is zero when one is located at $r = 0$.

Hence,

$$G \ = \ G(R) \ = \ \frac{G_N}{1 + \alpha\ G_N\ k^2} \ = \ \frac{G_N\ D(R)^2}{D(R)^2 + \alpha\beta^2\ G_N}, \qquad (1.7)$$

such that $G(R(r = 0)) = G(R = 2G_N M_o) = 0$ consistent with the findings [16] since $D(R(r = 0)) = D(R = 2G_N M_o) = 0$.

An important remark is in order. There is a $fundamental\ difference$ between the work of Reuter et al [16] and ours . The metric components studied by [16] were of the form, $g_{tt} = 1 - 2G(r)M_o/r, ...$ and are *not* solutions of Einstein's field equations. Whereas in our case, the metric components (1.1) $g_{tt} = 1 - 2G(r)M(r)/r = 1 - 2G_N M_o/R(r), ...$ *are* solutions of Einstein' equations displayed in eq-(1.1). This is one of the most salient features in working with the most general metric (1.1) involving the radial functions $R(r)$ instead of forcing $R(r) = r$.

Hence, given that $R = R(r)$, by imposing the following conditions valid for *all* values of r

$$(1 - \frac{2\ G_N\ M_o}{R(r)}) \ = \ (1 - \frac{2\ G(r)\ M(r)}{r}). \qquad (1.8)$$

$$\frac{(\frac{dR}{dr})^2}{(1 - \frac{2\ G_N\ M_o}{R(r)})} \ = \ \frac{1}{(1 - \frac{2\ G(r)\ M(r)}{r})}. \qquad (1.9)$$

from eqs-(1.7, 1.8, 1.9) one infers that

$$\frac{dR}{dr} \ = 1 \ \Rightarrow R(r) \ = \ r + 2G_N M_o. \qquad (1.10)$$

which is the Brillouin choice for the radial function as well as the relation

$$G(r) \ = G_N \ (\frac{r}{R}) \ (\frac{M_o}{M(r)}) \ = \ (\frac{G_N \ D(R)^2}{D(R)^2 + \alpha\beta^2 \ G_N}) \ \Rightarrow$$

$$M(r) \ = \ M_o \ (\frac{r}{R}) \ (\frac{D(R)^2 + \alpha\beta^2 \ G_N}{D(R)^2}). \qquad (1.11)$$

that allows us to determine the form of the $M(r)$ once the radial function $R(r) = r + 2G_N M_o$ is plugged into $D(R)$ given by eq-(1.6). The constant found by Reuter et al [16] is $\alpha\beta^2 = 118/15\pi$ and the proper distance $D(R)$ is given by eq-(1.6).

When $r = 0$ a careful analysis reveals

$$M(r \to 0) \to \ (constant) \ \frac{1}{2 \ G_N \ M_o}.. \qquad (1.12)$$

therefore, the running mass parameter at $r = 0$, $M(r = 0) \sim 1/R(r = 0) = 1/(2G_N M_o)$ is $finite$ instead of being infinite. The running mass at $r = 0$ has a cutoff given by the inverse of the ultra-violet cutoff $R(r = 0) = 2G_N M_o$ (up to a numerical constant). When $r = 0$ one has in eqs-(1.7, 1.11) that $G(r = 0) = 0$. When $r \to \infty$ one has $M(r \to \infty) \to M_o$ as expected, where M_o is the Kepler mass observed by an observer at asymptotic infinity (deep infrared) and $G(r \to \infty) \to G_N$.

The running flow $M(r)$ was never studied by [16]. Our ansatz in eqs-(1.8, 1.9) is an heuristic one. In the special case when $M(r = 0) = M_o$ one gets the interesting result for the value of M_o given by $M_o \sim M_{Planck}$ which is the same, up to a trivial numerical factor, to the Planck mass remnant in the final state of the Hawking black hole evaporation process found by [16] after a Renormalization Group improvement of the Vaidya metric was performed.

Concluding, $R = r + 2G_N M_o$ is the sought after relation between r and R, out of an infinite number of possible functions $R(r)$ obeying the SSS vacuum solutions of Einstein's equations. We may notice that $r = r(R) = D(R)$ given by eq-(1.6) is the appropriate choice for the radial function if, and only if, the *spatial* area coincides with the *proper* area $4\pi R(r)^2$. The spatial area $A(r)$ is determined in terms of the infinitesimal spatial volume $dV(r)$ as follows :

$$dV(r) = A(r)dr \ \Rightarrow \ A(r) = 4\pi R(r)^2 \ \frac{(dR/dr)}{\sqrt{1 - 2G_N M_o/R(r)}}. \qquad (1.13a)$$

When $A(r) = 4\pi R^2$ then

$$\frac{(dR/dr)}{\sqrt{1 - 2G_N M_o/R}} \ = \ 1. \qquad (1.13b)$$

since the integration of eq-(1.12) was performed in eq-(1.6), one can infer then that $r = r(R) = D(R)$ is the choice in this case for the functional relationship between R and r; in particular $A(r = 0) = 4\pi(2G_N M_o)^2$, which is not true in general when the *proper* area is not equal to the *spatial* area. The volume is *zero* at $r = 0$.

To finalize this subsection, when the radial function $R = r + 2G_N M_o$ has been specified by the RG flow solutions [16] , the scalar curvature is

$$\mathcal{R} \ = \ - \frac{2\ G_N\ M_o\ \delta(r)}{R^2\ (dR/dr)} \ = \ - \frac{2\ G_N\ M_o\ \delta(r)}{(r + 2G_N M_o)^2}. \qquad (1.14a)$$

and has a delta function singularity at $r = 0$ of the form

$$-\frac{2\ G_N\ M_o\ \delta(r=0)}{(2G_N M_o)^2} \ = \ - \frac{\delta(r=0)}{2G_N M_o}. \qquad (1.14b)$$

compared to the *stronger* singular behaviour of the Hilbert textbook solution at $r = 0$ when $R = r$

$$\mathcal{R}(Hilbert) \ = \ - \frac{2\ G_N\ M_o\ \delta(r)}{R^2\ (dR/dr)} \ = \ - \frac{2\ G_N\ M_o\ \delta(r)}{r^2} \ \Rightarrow$$

$$\mathcal{R}(r=0) \ = \ - \frac{2\ G_N\ M_o\ \delta(r=0)}{0^2}. \qquad (1.14c)$$

The reason the singularity of (1.14b) is softer than in (1.14c) is because when there is an ultra-violet cutoff of the radial function $R(r = 0) = 2G_N M_o$ (due to the presence of matter) the proper area $4\pi(2G_N M_o)^2$ is finite at $r = 0$ and so is the surface mass density. However, since the *volume* is *zero* at the location $r = 0$ of the point-mass, the volume mass density is *infinite* and one cannot eliminate the singularity at $r = 0$ given by $\mathcal{R} = -\delta(r = 0)/(2G_N M_o)$.

1.3 Anti de Sitter-Schwarzschild Metrics and running Cosmological Constant

We begin with the generalized de Sitter and Anti de Sitter metrics that will help us understand the nature of the infrared cutoff required to solve the cosmological constant problem. In [10] we proved why the most general *static* form of the (Anti) de Sitter-Schwarzschild solutions are given in terms of an arbitrary radial function by

$$g_{00} = (\ 1 - \frac{2G_N M_o}{R(r)} - \frac{\Lambda_o}{3}\ R(r)^2\ ), \quad g_{rr} = -(\ 1 - \frac{2G_N M_o}{R(r)} - \frac{\Lambda_o}{3}\ R(r)^2\ )^{-1}\ (dR(r)/dr)^2. \qquad (1.15)$$

The angular part is given as usual in terms of the solid angle by $-(R(r))^2(d\Omega)^2$.

Λ_o is the cosmological constant. The $\Lambda_o < 0$ case corresponds to Anti de Sitter-Schwarzschild solution and $\Lambda_o > 0$ corresponds to the de Sitter-Schwarzschild solution. The physical interpretation of these solutions is that they correspond to "black holes" in curved backgrounds that are not asymptotically flat. For very small values of R one recovers the ordinary Schwarzschild solution. For very large values of R one recovers asymptotically the (Anti) de Sitter backgrounds of constant scalar curvature.

Since the radial function $R(r)$ can be arbitrary, one particular expression for the radial function $R(r)$, out of an *infinite* number of arbitrary expresions, in the de Sitter-Schwarzschild ($\Lambda_o > 0$) case one may choose [10]

$$\frac{1}{R - (2G_N M_o)} = \frac{1}{r} + \sqrt{\frac{\Lambda_o}{3}}. \qquad (1.16)$$

When $\Lambda_o = 0$ one recovers $R = r + (2G_N M_o)$ that has a similar behaviour at $r = 0$ and $r = \infty$ as the original Schwarzschild solution of 1916 given by $R^3 = r^3 + (2G_N M_o)^3$; i.e. $R(r = 0) = 2G_N M_o$ and $R(r \to \infty) \sim r$ respectively. When $M_o = 0$ one recovers the pure de Sitter case and the radial function becomes

$$\frac{1}{R} = \frac{1}{r} + \sqrt{\frac{\Lambda_o}{3}}. \qquad (1.17)$$

In this case, one encounters the *reciprocal* situation (the "dual" picture) of the Schwarzschild solutions : (i) when r tends to zero (instead of $r = \infty$) the radial function behaves $R(r \to 0) \to r$; in particular $R(r = 0) = 0$ and (ii) when $r = \infty$ (instead of $r = 0$) the value of $R(r = \infty) = R_{Horizon} = \sqrt{\frac{3}{\Lambda_o}}$ and one reaches the location of the *horizon* given by the condition $g_{00}[R(r = \infty)] = 0$.

A reasonable and plausible argument as to why the cosmological constant is not *zero* and why it is so *tiny* was given by [10] : In the pure de Sitter case, the condition

$$g_{00}(r = \infty) = 0 \ \Rightarrow \ 1 - \frac{\Lambda_o}{3} \ R(r = \infty)^2 = 0 \qquad (1.18)$$

has a real valued solution

$$R(r = \infty) = \sqrt{\frac{3}{\Lambda_o}} \ = R_{Horizon} \ = \ Infrared\ cutoff. \qquad (1.19)$$

and the correct order of magnitude of the observed cosmological constant can be derived from eq-(1.19) by equating $R(r = \infty) = R_{Horizon}$ = Hubble Horizon radius as seen today since the Hubble radius is *constant* in the very *late time* pure inflationary de Sitter phase of the evolution of the universe when the Hubble parameter is constant H_o. The metric in eq-(1.15) is the *static* form of the generalized de Sitter (Anti de Sitter) metric associated with a *constant* Hubble parameter.

Therefore, by setting the Hubble radius to be of the order of $10^{61}\ L_{Planck}$ and by setting $G = L^2_{Planck}$ ($\hbar = c = 1$ units) in

$$8\pi \ G \ \rho_{vacuum} \ = \ \Lambda_o \ = \ \frac{3}{R(r = \infty)^2} = \frac{3}{R_H^2} \Rightarrow$$

$$\rho_{vacuum} = \frac{3}{8\pi} \ \frac{1}{L_P^2} \ \frac{1}{R_H^2} = \frac{3}{8\pi} \ \frac{1}{L_P^4} \ (\frac{L_P}{R_H})^2 \ \sim \ 10^{-123} \ (M_{Planck})^4, \quad . \qquad (1.20)$$

we obtain a result which agrees with the experimental observations when $R_{Hubble} \sim 10^{61} L_{Planck}$.

Notice the importance of using the radial function $R = R(r)$ in eq-(1.17). Had one used $R = r$ in eq-(1.17) one would have obtained a *zero* value for the cosmological constant when $r = \infty$. Thus, the presence of the radial function $R(r)$ is essential to understand *why* the cosmological constant is not *zero* and why it is so *tiny* .

The idea now is to relate the metric components in the Anti de Sitter-Schwarzcshild case involving the *running* $G(r), M(r), \Lambda(r)$ parameters with the metric components of (1.15) involving the unique and sough-after radial function $R(r)$ and the constants G_N, M_o, Λ_o (as seen by an asymptotic observer in the deep infrared region). The equations which determine the forms of $M(r)$ and $R(r)$ are given by

$$(1 - \frac{2 \ G_N \ M_o}{R(r)} - \frac{\Lambda_o}{3} \ R(r)^2 \ ) \ = \ (1 - \frac{2 \ G(r) \ M(r)}{r} - \frac{\Lambda(r)}{3} \ r^2 \ ). \tag{1.21}$$

$$(1 - \frac{2 \ G_N \ M_o}{R(r)} - \frac{\Lambda_o}{3} \ R(r)^2 \ )^{-1} \ (dR(r)/dr)^2 \ = \ (1 - \frac{2G(r) \ M(r)}{r} - \frac{\Lambda(r)}{3} \ r^2 \ )^{-1}. \tag{1.22}$$

then from eqs-(1.21, 1.22) one infers that

$$\frac{dR}{dr} = 1 \ \Rightarrow R(r) \ = \ r + \ R_* \tag{1.23}$$

where the constant of integration R_* is now the root of the cubic equation, and *not* the value $2G_N M_o$, given by

$$1 - \frac{2 \ G_N \ M_o}{R(r=0)} + \frac{\Lambda_o}{3} \ R(r=0)^2 \ = \ 1 - \frac{2 \ G_N \ M_o}{R_*} + \frac{\Lambda_o}{3} \ R_*^2 \ = \ = 0. \tag{1.24}$$

such that $g_{tt}(R(r=0)) = g_{tt}(R_*) = 0$. The real positive root of the cubic equation (found after multiplying (1.24) by $R_* \neq 0$) is

$$R_* \ = \ [\ \frac{3G_N M_o}{|\Lambda_o|} + \sqrt{ \frac{(3G_N M_o)^2}{\Lambda_o^2} + \frac{1}{|\Lambda_o|^3} } \ ]^{1/3} \ + \ [\ \frac{3G_N M_o}{|\Lambda_o|} - \sqrt{ \frac{(3G_N M_o)^2}{\Lambda_o^2} + \frac{1}{|\Lambda_o|^3} } \ ]^{1/3}. \tag{1.25}$$

Because Anti de Sitter space has $\Lambda_{AdS} < 0$, we have already aken into account the negative sign in the expression in eq-(1.25) by writing $\Lambda_{AdS} = -|\Lambda_o|$ and we must disregard the two complex roots (a pair of complex conjugates).

The values of R range from $0 \ < \ R_* \ \leq \ R \ \leq \ \infty$ and correspond to the values of r ranging from $0 \ \leq \ r \ \leq \ \infty$. This is very reasonable since R has an ultra-violet cutoff given by the root of the cubic $R_* > 0$. If R was allowed to attain the values of *zero* the metric component g_{tt} would blow up. r can in fact attain the *zero* value, but not the radial function $R(r) = r + R_*$. The metric component g_{rr} in (1.15) blows up at $r = 0$, location of the singularity.

Notice that one cannot take the limits $\Lambda_0 \rightarrow 0$ in eq-(1.25) *after* having found the roots of the cubic equation because that limit is singular. One must take the limit $|\Lambda_o| \rightarrow 0$ of eq-(1.24) before and afterwards find the root of $g_{tt}(R_*) = 0$ given by $R_* = 2G_N M_o$ (when $|\Lambda_o| = 0$).

After having found the root R_* of the cubic equation, from eq-(1.21) one infers

$$\frac{2\ G_N\ M_o}{r\ +\ R_*}\ +\ \frac{\Lambda_o}{3}\ (r+R_*)^2\ =\ \frac{2\ G(r)\ M(r)}{r}\ +\ \frac{\Lambda(r)}{3}\ r^2. \qquad (1.26)$$

which yields $M(r)$

$$M(r)\ =\ \frac{r}{2G(r)}\ [\ \frac{2G_N M_o}{r+R_*}\ +\frac{\Lambda_o}{3}\ (r+R_*)^2 -\ \frac{\Lambda(r)}{3}\ r^2\ ]. \qquad (1.27)$$

where now the proper distance $D(R)$ associated with the metric (1.15) is given the *elliptic* integral whose lower limit of integration is now given by the cubic root R_* (instead of $2G_N M_o$) :

$$D(R) = \int_{R_*}^{R} \sqrt{g_{RR}}\ dR\ =\ \int_{R_*}^{R} \frac{dR}{\sqrt{1-\frac{2\ G_N\ M_o}{R}\ +\frac{|\Lambda_o|}{3}\ R^2}}\ =\ Elliptic\ Integral. \qquad (1.28a)$$

such

$$D(R(r=0)) = D(R=R_*) = 0. \qquad (1.28b)$$

The running coupling is the one given by [16]

$$G\ =\ G(R)\ =\ \frac{G_N}{1+\alpha\ G_N\ k^2}\ =\ \frac{G_N\ D(R)^2}{D(R)^2+\alpha\beta^2\ G_N}. \qquad (1.29)$$

where $D(R)$ is given by the elliptic integral and the running cosmological parameter is [16]

$$|\Lambda(k)|\ =\ |\Lambda_o|\ +\frac{b\ G_N}{4}\ (k^4)\ =\ |\Lambda_o|\ +\frac{b\ G_N}{4}\ \frac{\beta^4}{D(R)^4}. \qquad (1.30)$$

where the momentum-scale relation is $k^2\ =\ (\beta^2/D(R)^2)$.

As expected, in eq-(1.27) we have the correct limits : $M(r\to\infty)\to M_o$, since when $r\to\infty$, $R(r)\to r$, $|\Lambda(r)|\to|\Lambda_o|$ and $G(r)\to G_N$. $M(r=0)\sim 1/R_*$ is finite also because $r/G(r)$ and $\Lambda(r)r^2$ are *finite* as $r\to 0$.

In the case of de Sitter-Schwarzschild metric , $\Lambda_o > 0$, one has a negative real root and a positive double root [10] $R_2 = R_3 > 0,\ R_1 < 0$; however, there is *no* horizon since g_{tt} does not change signs as once crosses the double-root location ; there is problem with the $R_1 < 0$ solutions and there is a pole of g_{tt} at $R = 0$. For this reason we have focused on the Anti de Sitter-Schwarzschild metric in this subsection.

2 Jordan-Brans-Dicke Gravity

We wish now to relate the metric of eq-(1.1) that solves the vacuum Einstein field equations for $r > 0$ written in terms of $G_N, M_o, R(r)$ with a metric written in terms of $G(r), M(r), r$

that does *not* solve the vacuum field equations but instead the field equations in the presence of a scalar field Φ associated with the Jordan-Brans-Dicke theory of gravity. Such metric is given by

$$(ds)^2 = g_{tt}(r)\ (dt)^2\ - g_{rr}(r)\ (dr)^2\ - \rho(r)^2\ (d\Omega)^2. \tag{2.1}$$

A conformal transformation $g'_{\mu\nu}\ = e^{2\lambda}\ g_{\mu\nu}$ relating the two metrics can be attained by starting with the Brans-Dicke-Jordan scalar-tensor action

$$\int\ d^4x\ \sqrt{g}\ [\ \Phi^2\ \mathcal{R}\ +\ 6\ (\nabla_\mu\Phi)\ (\nabla^\mu\Phi)\ ]. \tag{2.2}$$

and which can be transformed into a pure gravity action by means of a conformal transformation

$$g'_{\mu\nu}\ =\ e^{2\lambda}\ g_{\mu\nu};\quad \sqrt{g'}\ =\ e^{4\lambda}\ \sqrt{g}. \tag{2.3}$$

$$\sqrt{g'}\ \mathcal{R}'(g')\ =\ \sqrt{g}\ e^{2\lambda}\ [\ \mathcal{R}\ - 6\ (\nabla_\mu\nabla^\mu\ \lambda)\ - 6\ (\nabla_\mu\lambda)(\nabla^\mu\lambda)\ ] \tag{2.4}$$

By setting

$$e^{2\lambda}\ \equiv\ \frac{\Phi^2}{\Phi_o^2}\ =\ \frac{G_N}{G(r)}. \tag{2.5}$$

one can rewrite :

$$\sqrt{g'}\ \mathcal{R}'(g')\ =\ \frac{\sqrt{g}}{\Phi_o^2}\ [\ \Phi^2\mathcal{R}\ - 6\ \Phi\ (\nabla_\mu\nabla^\mu\ \Phi)\ ] \tag{2.6}$$

due to the fact that $(\nabla_\mu\ \sqrt{g}) = 0$ then

$$\sqrt{g}\ \Phi(\nabla_\mu\nabla^\mu\ \Phi) = \nabla_\mu\ (\sqrt{g}\ \Phi\nabla^\mu\ \Phi) - \sqrt{g}\ (\nabla_\mu\Phi)(\nabla^\mu\Phi). \tag{2.7}$$

since total derivative term drops from the action one has the equalities

$$\int\ d^4x\ \sqrt{g}\ [\ \Phi^2\ \mathcal{R}\ +\ 6\ (\nabla_\mu\Phi)\ (\nabla^\mu\Phi)\ ]\ =\ \int\ d^4x\ \sqrt{g}\ [\ \Phi^2\ \mathcal{R}\ - 6\ \Phi\ (\nabla_\mu\nabla^\mu\ \Phi)\ ]\ =$$

$$\frac{1}{16\pi G_N}\ \int\ d^4x\ \sqrt{g'}\ \mathcal{R}'(g') \tag{2.8}$$

therefore, one can solve the Einstein vacuum field equations for the metric $g'_{\mu\nu}$ (for $r > 0$) and perform a conformal transformation $g'_{\mu\nu}\ = e^{2\lambda}\ g_{\mu\nu}$ to obtain the metric that solves the field equations corresponding to the Jordan-Brans-Dicke action.

The running Newtonian coupling $G(r)$ is now defined explicitly in terms of the scalar field as follows

$$\Phi^2 = \frac{1}{16\pi G(r)};\quad \Phi_o^2 = \frac{1}{16\pi G_N}. \tag{2.9a}$$

and the dimensionless scaling factor $e^{2\lambda}$ is given by the ratio :

$$e^{2\lambda} = \frac{G_N}{G(r)} = \frac{\Phi^2}{\Phi_o^2}. \tag{2.9b}$$

such that the equalities among the three lines of eq-(2.5) are satisfied.

The scalar field Φ that determines the functional form of $G(r)$ must solve the generalized Klein-Gordon equation obtained from a variation of the action (2.5) w.r.t the scalar field Φ

$$(\nabla_\mu \nabla^\mu \ - \frac{1}{6}\mathcal{R})\ \Phi \ = 0, \quad for\ r > 0. \tag{2.10}$$

and the latter equation is *equivalent* to the equation $\mathcal{R}'(g') = 0$ since the scalar curvature $\mathcal{R}$, for $r > 0$, is fixed by eq-(2.6) after setting $\mathcal{R}'(g') = 0$ because the metric $g'_{\mu\nu}$ is a solution of the Einstein vacuum field equations for $r > 0$. When $\mathcal{R}'(g') = 0$, for $r > 0$, yields the scalar curvature

$$\mathcal{R}(g) \ = \ \frac{6}{\Phi}\ (\nabla_\mu \nabla^\mu\ \Phi). \tag{2.11}$$

which is precisely *equivalent* to the generalized Klein-Gordon equation (2.10). This means that the scalar Φ field does not have dynamical degrees of freedom since it is identified with the conformal factor $e^\lambda = \Phi/\Phi_o$. Therefore one can safely equate the scalar field Φ^2 with $(1/16\pi G(r))$ giving

$$\mathcal{R}(r) \ = \ \frac{6}{\Phi}\ (\nabla_r \nabla^r\ \Phi) \ = $$

$$\frac{6}{\sqrt{G(r)}}\ \frac{1}{\sqrt{g}}\ \partial_r\ (\ \sqrt{g}\ g^{rr}\ \partial_r\ \sqrt{G(r)}\ ). \tag{2.12}$$

where the metric components $g_{\mu\nu}$ necessary to evaluate the Laplace-Beltrami operator are obtained directly via the conformal scaling of the metric that solves the vacuum static spherical solutions of Einstein's equations of the previous section :

$$g_{tt} \ = \ e^{-2\lambda}\ (1 - \ \frac{2G_N M_o}{R(r)}). \tag{2.13}$$

$$g_{RR} \ = \ e^{-2\lambda}\ \frac{1}{1 - \ \frac{2G_N M_o}{R}}, \quad g_{rr} \ = \ g_{RR}\ (dR/dr)^2. \tag{2.14}$$

$$g_{\phi\phi} \ = \ e^{-2\lambda}\ R(r)^2 \ = \ \rho(r)^2, \quad g_{\theta\theta} \ = \ e^{-2\lambda}\ R(r)^2\ sin^2(\phi). \tag{2.15}$$

$$\sqrt{g} \ = \ e^{-4\lambda}\ R(r)^2\ (dR/dr)\ sin(\phi). \tag{2.16}$$

Since $e^{-2\lambda} = G(r)/G_N$ and $G(r = 0) = 0$ then the radial rho function obeys the condition $\rho(r = 0) = 0$.

The new proper distance $D(R)$ is now given by

$$D(R) \;=\; \int_{2G_N M_o}^{R} \frac{e^{-\lambda}}{\sqrt{1-\;(2G_N M_o/R)}}\, dR \;=\; \int_{2G_N M_o}^{R} \frac{(G(R)/G_N)^{1/2}}{\sqrt{1-\;(2G_N M_o/R)}}\, dR \quad (2.17)$$

and *differs* from the expressions of eq-(1.6) because of the conformal factor.

However, there is a *caveat* if we now try to use the running flow of the Newtonian coupling of the previous section [16]

$$G(R) = \frac{G_N\ D(R)^2}{D(R)^2 + \alpha\beta^2\ G_N} \;\Rightarrow\; D(R) \;=\; \sqrt{\frac{(\alpha\beta^2\ G_N)\ G(R)}{G_N\ -\ G(R)}}. \quad (2.18)$$

because the RG flow equations *must differ* now due to the *presence* of the scalar field Φ. To prove why one *cannot* use the running flow equation (2.18) for G used in section **1.2, 1.3**, let us differentiate both sides of the expression for $D(R)$ in eq-(2.18) and upon equating the result with the integrand of eq-(2.17) leads to the *differential* equation obeyed by $G(R)$:

$$\frac{dD(R)}{dR} \;=\; \frac{\alpha\beta^2\ G_N^2}{2\ (G_N - G(R))^2\ \sqrt{\frac{(\alpha\beta^2 G_N)\ G(R)}{G_N\ -\ G(R)}}}\ \frac{dG(R)}{dR} \;=\; \frac{(G(R)/G_N)^{1/2}}{\sqrt{1-\;(2G_N M_o/R)}}. \quad (2.19)$$

subject to the boundary conditions $G(R(r=0)) = G(R = 2G_N M_o) = 0$ and $G(r \to \infty) = G(R \to \infty) \to G_N$. The differential equation (2.19) *is* the equation that determines the functional form of $G(R)$. Notice that functional form of $G(R)$ which obeys the above differential equation is *not* the same as the result obtained for $G(R)$ in eq-(1.7) of the previous section because the proper distance $D(R)$ given by the integral of eq-(2.17) *differs* from the integral of eq-(1.6). The constant found by Reuter et al [16] is $\alpha\beta^2 = 118/15\pi$.

One can integrate eq-(2.19) giving the functional relationship between G and R :

$$\frac{\sqrt{\alpha\beta^2}\ G_N^2}{2} \int_{G_o}^{G} \frac{dG}{G\ \sqrt{(G_N - G)^3}} \;=\; \int_{2G_N M_o}^{R} \frac{dR}{\sqrt{1-\;(2G_N M_o/R)}} \;=$$

$$\frac{\sqrt{\alpha\beta^2}\ G_N^2}{2}\ [\ \frac{2}{G_N\sqrt{(G_N - G)}} \;-\; \frac{2\ arctanh\ [\ \sqrt{1-(G/G_N)}\ ]}{(G_N)^{3/2}}\ ] \;-\; I[G_o] \;=$$

$$\sqrt{R\ (\ R - 2\ G_N\ M_o)} \;+\; 2\ G_N\ M_o\ ln\ [\ \sqrt{\frac{R}{2G_N M_o}} \;+\; \sqrt{\frac{R - 2G_N M_o}{2G_N M_o}}\ ]. \quad (2.20)$$

where $G_o \equiv G(R = 2G_N M_o)$.

One can immediately deduce that the first integral *diverges* when $G = G_N$ which is compatible with the condition $G(R \to \infty) = G_N$. But there is a problem in enforcing the behaviour of $G(r=0) = 0$; one cannot impose the condition $G_o \equiv G(R = 2G_N M_o) = 0$ because the G integral also *diverges* when $G = G_o = 0$! (the integral is $-\infty$).

Therefore, one must have the condition $G_o \equiv G(R = 2G_N M_o) \neq 0$. The value of G_o obeying $G_N > \ G_o = G(r = 0) \ > \ 0$ can be determined from solving the transcendental equation derived from the condition

$$I[G_o] \ = \ \frac{\sqrt{\alpha\beta^2} \ G_N^2}{2} \ [\ \frac{2}{G_N \sqrt{(G_N - G_o)}} \ - \ \frac{2 \ arctanh \ [\ \sqrt{1 - (G_o/G_N)} \ ]}{(G_N)^{3/2}} \ ] \ = \ 0. \quad (2.21)$$

The result $I[G_o] = 0$ is now compatible with the behaviour of the R integral which is *zero* when $R(r = 0) = 2G_N M_0$. To sum up : one *cannot* satisfy the condition $G(R(r = 0)) = 0$ required by eqs-(1.7, 2.18) found by [16].

The same conclusions apply (one is forced to impose $G_o > 0$) if we had taken a *minus* sign in front of the square root in the R integral which leads to $G(r \to \infty) = 0$ ($R \sim r$ when $r \to \infty$), as opposed to the desired behaviour $G(r \to \infty) \to G_N$ It is interesting that this result $G(r \to \infty) = 0$, when the minus sign in front of the square root is chosen, is "dual" to the behaviour found in the RG flow solutions by [16] where at $r = 0$ (instead of $r = \infty$) one encounters $G(R(r = 0)) = G(R = 2G_N M_o) = 0$ (asymptotic freedom).

Concluding, the fact that G integral (2.20) *diverges* at $G = 0$ is a signal that one *cannot* use the running flow equation (2.18) for G in the presence of the Jordan-Brans-Dicke scalar Φ. One would have to solve the *modified* RG equations that will involve the *beta* functions for the Φ field in addition to the metric $g_{\mu\nu}$. A similar *divergence* problem was encountered by [17]. One can bypass this divergence problem by imposing $G(r = 0) = G(R = 2G_N M_0) = G_o > 0$ where G_o is given by a solution of the trascendental equation. By taking the minus sign in front of the square root we found an ultraviolet/infrared "duality" behaviour of the couplings, at $r = 0$ and $R \sim r \to \infty$, which warrants further investigation.

3 Concluding Remarks : On Noncommutative and Finsler Geometries

We conclude by discussing some speculative remarks. It is well known (see references in [17]) that by replacing $G_N \to G(k^2) = G_N(1 + G_N k^2)^{-1}$ leads to $1/k^4$ modifications of the propagator

$$\frac{G(k^2)}{k^2} \ = \ \frac{G_N}{k^2 \ (1 + G_N k^2)} \ = \ G_N \ [\ \frac{1}{k^2} \ - \ \frac{1}{k^2 \ + \ M^2_{Planck}} \ ], \quad G_N M^2_{Planck} = 1. \quad (3.1)$$

that correspond to quadratic curvatures $\mathcal{R}^2$ of perturbative quantum gravity. The Lanczos-Lovelock theories of Gravity involving higher powers of the curvature have the attractive feature that the equations of motion are no more than second order in derivatives of the metric and contain no ghosts. The authors [18] have found black hole solutions, topological defects, and naked singularities as well, in pure Lanczos-Lovelock Gravity with only one Euler density term. The fact that naked singularities were found by [18] deserve further investigation within the context of modified propagators induced by a running Newtonian coupling.

Another interesting field of study is Noncommuttaive Geometry, Fuzzy spaces, Fractal geometries, etc... The standard noncommutative algebra (there are far more fundamental algebras like Yang's algebra in noncommutative phase spaces) is of the form

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}. \quad [p^\mu, p^\nu] = 0 \quad [x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{3.2}$$

where $\eta^{\mu\nu}$ is a flat space metric and the structure constants (c-numbers) $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$ are c-numbers that commute with x, p and that have dimensions of $length^2$; the $\Theta^{\mu\nu}$ are proportional to the L^2_{Planck}. A change of coordinates

$$x'^\mu = x^\mu + \frac{1}{2}\Theta^{\mu\rho}\ p_\rho. \quad p'^\mu = p^\mu. \tag{3.3}$$

leads to an algebra with commuting coordinates and momenta

$$[x'^\mu, x'^\nu] = 0. \quad [p'^\mu, p'^\nu] = 0. \quad [x'^\mu, p'^\nu] = i\eta^{\mu\nu}. \tag{3.4}$$

Due to the *mixing* of coordinates and momentum in the new commuting variables x', p' one can envisage coordinate and momentum dependent metrics in phase space, in particular Finsler geometries, and whose average over the momentum coordinates $< \pi_{\mu\nu}(x,p) >_p \ = \ g_{\mu\nu}(x)$ yield the effective spacetime metric. This momentum averaging procedure is very similar to the averaging of the momentum-scale dependent metrics employed in the Renormalization Group flow of the effective average action by [16]. Morever, the momentum dependence of the new coordinates x' leads to a momentum dependent radial coordinate $r' = \sqrt{x'^\mu\ x'_\mu}$ involving commuting x'^μ coordinates

$$r' \ = \ \sqrt{(x^\mu + \frac{1}{2}\Theta^{\mu\rho}\ p_\rho)\ (x_\mu + \frac{1}{2}\Theta_{\mu\tau}\ p^\tau)} \ \sim \ r\ [\ 1 + \frac{1}{4r^2}\ \Theta^{\mu\nu}\ x_\mu\ p_\nu + ...\ ]. \tag{3.5}$$

Similar attempts to study the Noncommutative effects on black holes by modifying $r \rightarrow r'$ have been made by many other authors, however, to our knowledge its relation to phase spaces and Finsler geometries has not been explored. The impending question is to find another interpretation of the radial function $R(r)$ and the physical meaning of the cutoff $R(r=0) = 2G_N M_o$ in terms of the momentum dependent radial coordinate r'.

When $x^\mu = 0 \Rightarrow r = 0$ and (3.5) becomes

$$r' \ = \ \frac{1}{2}\sqrt{\Theta^{\mu\rho}\ p_\rho\ \Theta_{\mu\tau}\ p^\tau}. \tag{3.6}$$

The expression inside the square root can be written in terms of $p_\mu p^\mu = M_o^2$ as

$$\Theta^{\mu\rho}\ \Theta_{\mu\tau}\ p_\rho\ p^\tau \ = \ (constant)^2\ L_P^4\ p_\mu\ p^\mu = (constant)^2\ L_P^4\ M_o^2. \tag{3.7a}$$

adjusting the value of the $constant = 4$, gives then the ultra-violet cutoff

$$r'(r=0) \ = \ \frac{1}{2}\sqrt{\Theta^{\mu\rho}\ p_\rho\ \Theta_{\mu\tau}\ p^\tau} \ = \ 2\ L_P^2\ M_o \ = \ 2\ G_N\ M_o. \tag{3.7b}$$

consistent with $R(r=0) = 2G_N M_o$ with the only subtlety that that $r = \sqrt{x^\mu x_\mu}$ involves now noncommuting coordinates x^μ.

When $r \neq 0$, the terms

$$\Theta^{\mu\rho} \ p_\rho \ x_\mu \ + \ \Theta_{\mu\tau} \ x^\mu \ p^\tau \ = \ \Theta^{\mu\rho} \ p_\rho \ x_\mu \ + \ \Theta^{\mu\tau} \ x_\mu \ p_\tau \ =$$

$$\Theta^{\mu\rho} \ p_\rho \ x_\mu \ + \ \Theta^{\mu\rho} \ x_\mu \ p_\rho \ = \ \Theta^{\mu\rho} \ (\ x_\mu \ p_\rho - i \ \eta_{\mu\rho} \ ) \ + \ \Theta^{\mu\rho} \ x_\mu \ p_\rho \ = \ 2 \ \Theta^{\mu\rho} \ x_\mu \ p_\rho. \quad (3.8)$$

due to the antisymmetric property of $\Theta^{\mu\rho}$, one has $\Theta^{\mu\rho} \ \eta_{\mu\rho} = 0$.

Becuase the quantity $\Theta^{\mu\rho} \ x_\mu \ p_\rho$ can be interpreted as a modified angular momentum operator, in the spherically symmetric case, by imposing the condition

$$\Theta^{\mu\rho} \ x_\mu \ p_\rho \ \sim \ L^2_{Planck} \ M_o \ \omega(r) \ r^2 \ = \ G_N \ M_o \ \omega(r) \ r^2 \quad (3.9)$$

where $\omega(r)$ is a scale-dependent frequency, one is able to express the commuting $r' = \sqrt{x'^\mu \ x'_\mu}$ variable in terms of the non-commuting one $r = \sqrt{x^\mu x_\mu}$:

$$r' = r'(r) \ = \ \sqrt{r^2 \ + \ 2\Theta^{\mu\rho} \ x_\mu \ p_\rho \ + \ (2G_N M_o)^2} \ =$$

$$\sqrt{r^2 \ + \gamma \ G_N M_o \ \omega(r) \ r^2 \ + \ (2G_N M_o)^2}. \quad (3.10)$$

where γ is a constant. The question is to see whether or not metrics $g_{\mu\nu}(R)$ expressed in terms of radial functions $R(r)$ of the *noncommuting* radial variable r solve the Noncommutative deformations of Einstein's equations; for example the field equations corresponding to Moyal star product deformations of Einstein Gravity [20]. When $r = 0$ one recovers the cutoff $r'(r=0) = 2G_N M_o$. Concluding, this procedure to relate the effects of the Noncommutativity of coordinates with the ultra-violet cutoff $R(r=0) = 2G_N M_o$ is quite promising . We shall leave it for future work.

Let us summarize the main conclusions of this work :

1. The original Schwarzschild's 1916 solution has *no horizons* and is *static* for *all* values of r with a timelike naked singularity at $r = 0$. The radial function $R = [r^3 + (2G_N M_o)^3]^{1/3}$ *has* an UV cutoff in $R(r=0) = 2G_N M_o$.

2. The "black hole" entropy expression is the same as the Euclideanized Einstein-Hilbert action corresponding to the scalar curvature delta function singularity due to the presence of a mass point at the origin $r = 0$. Such delta function scalar curvature singularity can *account* for the "black hole" entropy. For this reason a microscopic theory of a point-mass is needed to understand key aspects of Quantum Gravity. A point-mass may be *stringy* in Nature since due to the ultra-violet cutoff $R(r=0) = 2G_N M_o$, a point-mass source at $r = 0$ has non-zero area but zero volume; a string world-sheet has non-zero area and zero volume.

3. In sections **1.2, 1.3** we showed how the exact Nonperturbative Renormalization Group flow of the running Newtonian coupling $G(r)$ in Quantum Einstein Gravity [16] was consistent with the existence of an ultra-violet cutoff $R(r=0) = 2G_N M_o$

of the Schwarzschild solutions in eq-(1.1), after setting $g_{tt} = 1 - 2G_N M_o/R(r) = 1 - 2G(r)M(r)/r, ...$ We proved that due to the condition $G(r = 0) = 0$ and $M(r = 0) \sim 1/2G_N M_o$, there was *no* horizon since it is at the location $r = 0$ that $g_{tt}(r = 0) = 0$.

4. Similar results followed in the case of Anti de Sitter-Schwarzschild metrics with a running cosmological parameter $\Lambda(r)$ and Newtonian coupling $G(r)$. The ultra-violet cutoff in this case was no longer given by $2G_N M_o$ but instead by a real-valued positive root R_* of the cubic equation associated with the condition $g_{tt}(R(r = 0)) = g_{tt}(R_*) = 0$. There was a singularity at $r = 0$.

5. Generalized de Sitter metrics led to an *infrared* cuttoff $R(r = \infty) = R_{Hubble} = (3/\Lambda_o)^{1/2}$ in the very late time de Sitter inflationary phase of the evolution of the universe (when the Hubble parameter is *constant*) and provided a plausible argument why the cosmological constant is not *zero* and why it is so *tiny* [10].

6. In section **2** we studied how a running Newtonian coupling $G(r)$ could also be accommodated naturally in a Jordan-Brans-Dicke scalar-tensor theory of Gravity via a trivial conformal transformation of the Schwarzschild metric solution. However, the running Newtonian coupling $G(r) = (16\pi\Phi^2)^{-1}$ corresponding to the scalar field Φ could not satisfy the asymptotic freedom condition $G(r = 0) = 0$ found by [16]. Nevertheless, our results in section **2** exhibited an interesting ultra-violet/infrared *duality* behaviour of $G(r)$ that warrants further investigation. A combinatorial geometry and dual nature of gravity was proposed by [19] using Matroid theory.

To finalize we should stress the search for the foundational (quantum equivalence) principle of Quantum Gravity which is related to the true origin of inertia (mass/energy). Mach's principle is an intriguing concept with several formulations and applications [21], [23], [24], [25], [26], [22]. A proper and precise implementation of Mach's principle, beyond the equivalence's principle of General Relativity, in modern physics is still lacking, to our knowledge. Furthermore, it is very likely that our naive notions of Topology break down at small scales [13] and for this reason we must redefine our notion of a "point" such that this novel "fuzzy" topology is compatible with the *stringy geometry*. For the role of Fractals in the construction of a Scale Relativity theory based on scale resolutions of "points" and the minimal Planck scale see [15].

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Appendix A

Consider the conformal map

$$g'_{\mu\nu} = e^{2\lambda} g_{\mu\nu}. \qquad (A.1)$$

Here, the indices μ, ν run from $0, 1, ..., d-1$. The Christoffel symbols become

$$\Gamma'^{\mu}_{\alpha\beta}(g') = \Gamma^{\mu}_{\alpha\beta}(g) + \Sigma^{\mu}_{\alpha\beta}, \tag{A.2}$$

where

$$\Sigma^{\mu}_{\alpha\beta} = \delta^{\mu}_{\alpha}\lambda_{,\beta} + \delta^{\mu}_{\beta}\lambda_{,\alpha} - g_{\alpha\beta}\lambda^{,\mu}. \tag{A.3}$$

Using $(A.2)$ one finds that the Riemann tensor can be written as

$$\mathcal{R}'^{\mu}_{\nu\alpha\beta}(g') = \mathcal{R}^{\mu}_{\nu\alpha\beta}(g) + \nabla_{\alpha}\Sigma^{\mu}_{\nu\beta} - \nabla_{\beta}\Sigma^{\mu}_{\nu a} + \Sigma^{\mu}_{\sigma\alpha}\Sigma^{\sigma}_{\nu\beta} - \Sigma^{\mu}_{\sigma\beta}\Sigma^{\sigma}_{\nu\alpha} \tag{A.4}$$

where ∇_{α} denotes covariant derivative in terms of $\Gamma^{\mu}_{\alpha\beta}(g)$. By straightforward computation, using $(A.3)$ we find

$$\begin{aligned}\mathcal{R}'^{\mu}_{\nu\alpha\beta}(g') = \mathcal{R}^{\mu}_{\nu\alpha\beta}(g) + \{\delta^{\mu}_{\beta}\nabla_{\alpha}\lambda_{,\nu} - \delta^{\mu}_{\alpha}\nabla_{\beta}\lambda_{,\nu} - g_{\nu\beta}\nabla_{\alpha}\lambda^{,\mu} + g_{\nu\alpha}\nabla_{\beta}\lambda^{,\mu}\} \\ + \{(\delta^{\mu}_{\alpha}\lambda_{,\beta} - \delta^{\mu}_{\beta}\lambda_{,\alpha})\lambda_{,\nu} - (\delta^{\mu}_{\alpha}g_{\nu\beta} - \delta^{\mu}_{\beta}g_{\nu\alpha})\lambda_{,\sigma}\lambda^{,\sigma} - (g_{\nu\alpha}\lambda_{,\beta} - g_{\nu\beta}\lambda_{,\alpha})\lambda^{,\mu}\}.\end{aligned} \tag{A.5}$$

From $(A.5)$ we get the Ricci tensor

$$\begin{aligned}\mathcal{R}'_{\nu\beta}(g') = \mathcal{R}_{\nu\beta}(g) - \{(d-2)\nabla_{\beta}\lambda_{,\nu} + g_{\nu\beta}\nabla_{\mu}\lambda^{,\mu}\} \\ +(d-2)\{\lambda_{,\beta}\lambda_{,\nu} - g_{\nu\beta}\lambda_{,\mu}\lambda^{,\mu}\},\end{aligned} \tag{A.6}$$

which in turn gives us the scalar curvature

$$\mathcal{R}' = e^{-2\lambda}\{\mathcal{R} - 2(d-1)\nabla_{\mu}\lambda^{,\mu} - (d-2)(d-1)\lambda_{,\mu}\lambda^{,\mu}\}. \tag{A.7}$$

Therefore we get

$$\sqrt{-g'}\mathcal{R}' = \sqrt{-g}e^{(d-2)\lambda}\{\mathcal{R} - 2(d-1)\nabla_{\mu}\lambda^{,\mu} - (d-2)(d-1)\lambda_{,\mu}\lambda^{,\mu}\}. \tag{A.8}$$

Since $\nabla_{\mu}\sqrt{-g} = 0$, $(A.8)$ can also be written as

$$\begin{aligned}\sqrt{-g'}\mathcal{R}' = \sqrt{-g}e^{(d-2)\lambda}\mathcal{R} - \nabla_{\mu}\{(\tfrac{2(d-1)}{d-2})\sqrt{-g}(e^{(d-2)\lambda})^{,\mu}\} \\ +(d-2)(d-1)\sqrt{-g}e^{(d-2)\lambda}\lambda_{,\mu}\lambda^{,\mu}.\end{aligned} \tag{A.9}$$

We observe that the second term is a total derivative and therefore can be dropped. So, we have

$$\sqrt{-g'}\mathcal{R}' = \sqrt{-g}e^{(d-2)\lambda}(\mathcal{R} + (d-2)(d-1)\lambda_{,\mu}\lambda^{,\mu}). \tag{A.10}$$

For $d = 4$ the expression $(A.10)$ is reduced to

$$\sqrt{-g'}\mathcal{R}' = \sqrt{-g}e^{2\lambda}(\mathcal{R} + 6\lambda_{,\mu}\lambda^{,\mu}). \tag{A.11}$$

Some times it becomes convenient to write $e^{\lambda} = \Phi$. In this case, we have $\lambda,_{\mu} = \Phi^{-1}\Phi,_{\mu}$. Consequently, we see that (A.11) can also be written as

$$\sqrt{-g'}\mathcal{R}' = \sqrt{-g}(\Phi^2\mathcal{R} + 6\Phi,_{\mu}\Phi^{,\mu}) \quad (A.13)$$

or

$$\sqrt{-g'}\mathcal{R}' = \sqrt{-g}(\Phi^2\mathcal{R} + 6\nabla_{\mu}\Phi\nabla^{\mu}\Phi). \quad (A.14)$$

since $\nabla_{\mu}\Phi = \Phi_{,\mu}$.

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