Noether's amazing theorems

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Abstract

The theorems stated and proven by Emmy Noether are reviewed from a historical and mathematical perspective. It emphasizes the Lagrangian and variational or functional formalism, using basic tools of differential and integral analysis. Finally, some examples and applications in Theoretical Physics are indicated, and the intuitive meaning of both theorems is explained. Finally, possible generalizations and extensions of the theorem are suggested, as well as a mention of the formalism of jets and differential forms that makes it possible to generalize these theorems with a coordinate-free language.

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1 Introduction

The Newtonian formulation of the laws of Mechanics leaves many questions unresolved, in addition to the fact that its treatment is complicated when dealing with forces and accelerations (or moments), which are vector magnitudes. In the 19th century, Lagrange and Hamilton, along with other researchers, developed alternative formulations of Classical Mechanics, known as Analytical or Rational Mechanics, using mathematical procedures known today as variational analysis and functional calculus.

2 Action and lagrangians

In Lagrange's formulation, the fundamental object is the action integral of a function, today called Lagrangian (or Lagrangian density in the version of fields or continuous systems, although due to abuse of language, Lagrangian density is still called Lagrangian). The action is

$$S(q) = \int_M L dt \tag{1}$$

The Lagrangian L is a mathematical function that depends on variables $q(t)$ called generalized coordinates (which can be scalars, vectors, or even tensors, spinors, etc.).

2.1 First order lagrangians, $L = L(q, \dot{q})$

For a Lagrangian that depends on the generalized coordinates and their first-order time derivatives, one has:

$$\delta L(q, \dot{q}) = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \tag{2}$$

Using Leibniz's rule of derivation of a product, we can write this expression as follows:

$$\delta L(q, \dot{q}) = \frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q \tag{3}$$

or rearranging the terms

$$\delta L(q, \dot{q}) = E_1(L)\delta q + \frac{d}{dt}\left(p\delta q\right). \tag{4}$$

where we have defined the generalized moment

$$p = \frac{\partial L}{\partial \dot{q}} \tag{5}$$

and the Euler operator of the first order

$$E_1(L(q, \dot{q})) = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \tag{6}$$

The motion of a body or system defined by generalized coordinates is that which minimizes the action (more generally, extremizes it), and $\delta S = 0$ generally implies that $\delta L = 0$ for arbitrary variations of $\delta what$. In fact, the minimization of the action is invariant except for boundary terms, that is, the minimization of the action (or extremization) implies that a total derivative term can be added to the Lagrangian, which only contributes with a constant and does not affect the equations of motion.

The fact that the Lagrangian is quasi-invariant (invariant except for a total time derivative) is usually expressed as the gauge principle

$$\delta L = \frac{d\Lambda}{dt} \tag{7}$$

and then the global variation of the first-order Lagrangian can be written

$$\delta L = \frac{\partial L}{\partial q}\delta q - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) = E_1(L) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) \tag{8}$$

$$\delta L = E_1(L) + \frac{d}{dt}(p\delta q) = \frac{d\Lambda}{dt} \tag{9}$$

o bien

$$\delta L = E_1(L)\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) = E_1(L) + \frac{d}{dt}(p\delta q - \Lambda) \tag{10}$$

The criticality of the action and the Lagrangian of the first order, implies the fulfillment of the Euler-Lagrange equations:

$$E_1(L) = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \tag{11}$$

Furthermore, the boundary term is the conserved quantity or Noether charge under arbitrary δq symmetry transformations (as we will see later, it is one of Noether's theorems):

$$C = \left(\frac{\partial L}{\partial \dot{q}}\delta q - \Lambda\right)$$

The funny thing about all of this is that we can generalize it to higher-order and arbitrary-high-order derivatives. Or even consider more general transformations, for example some that include the change of the temporal coordinates in addition to the fields in the spatial coordinates.

2.2 Second order lagrangians, $L(q, \dot{q}, \ddot{q})$

Now suppose that $L = L(q, \dot{q}, \ddot{q})$, which corresponds to a Lagrangian that depends on generalized position, velocity, and acceleration. The variation of the Lagrangian is now

$$\delta L(q, \dot{q}, \ddot{q}) = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta \ddot{q} \tag{12}$$

or, again using the Leibniz product rule

$$\delta L(q, \dot{q}, \ddot{q}) = \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) + \frac{\partial L}{\partial \ddot{q}}\delta \ddot{q} \tag{13}$$

For the last term on the right-hand side, we apply the Leibniz rule twice again to eliminate the temporal dependence of the variations as far as we can:

$$\frac{\partial L}{\partial \ddot{q}}\delta \ddot{q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\delta \dot{q}\right) - \left[\frac{d}{dt}\left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\delta \dot{q}\right) - \frac{d^2}{dt^2}\frac{\partial L}{\partial \dot{q}}\delta q\right] \tag{14}$$

Using the same type of rearrangement as with the first-order Lagrangian, we now obtain the total variation of the Lagrangian

$$\delta L = \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}\right)\delta q + \frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}}\delta q - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\right)\delta q + \frac{\partial L}{\partial \ddot{q}}\delta \dot{q}\right] \tag{15}$$

This expression can be succinctly rewritten as

$$\delta L = E_2(L(q,\dot q,\ddot q)\delta q + \frac{d}{dt}\left[E_1(L(\dot q,\ddot q))\delta q + E_0(L(\ddot q))\delta\dot q\right] \tag{16}$$

and where we have defined the Euler operators

$$E_2(L(q,\dot q,\ddot q) = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot q} \tag{17}$$

$$E_1(L(\dot q,\ddot q)) = \frac{\partial L}{\partial \dot q}\delta q - \frac{d}{dt}\frac{\partial L}{\partial \ddot q} \tag{18}$$

$$E_0(L(\ddot q) = \frac{\partial L}{\partial \ddot q} \tag{19}$$

The extremization of the action (invariance) and the quasi-invariance of the Lagrangian generate the equations of motion

$$E_2 = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot q} = 0 \tag{20}$$

and the conservation of the Noether charge at the boundary

$$C = \left(\frac{\partial L}{\partial \dot q} - \frac{d}{dt}\frac{\partial L}{\partial \ddot q} - \Lambda\right)\delta q + \frac{\partial L}{\partial \ddot q}\delta\dot q \tag{21}$$

2.3 Third order lagrangians, $L = L(q,\dot q,\ddot q,\dddot q)$

It is left as an exercise for the interested reader and fascinated by these lines, to calculate the details (by brute force of derivation via Leibniz's rule) for the Lagrangian whose variation is

$$\delta L = \delta L(q,\dot q,\ddot q,\dddot q) = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta\dot q + \frac{\partial L}{\partial \ddot q}\delta\ddot q + \frac{\partial L}{\partial \dddot q}\delta\dddot q \tag{22}$$

Obviously, the problem is to rewrite the term

$$\frac{\partial L}{\partial \dddot q}\delta\dddot q \tag{23}$$

as follows

$$\frac{\partial L}{\partial \dddot q}\delta\dddot q = \frac{d}{dt}\left(\frac{\partial L}{\partial \dddot q}\delta\ddot q\right) - \frac{d}{dt}\left(\frac{d}{dt}\frac{\partial L}{\partial \dddot q}\delta\dot q\right) + \frac{d}{dt}\left(\frac{d^2}{dt^2}\frac{\partial L}{\partial \dddot q}\delta q\right) - \frac{d^3}{dt^3}\frac{\partial L}{\partial \dddot q}\delta q \tag{24}$$

Obviously, the problem is to rewrite the term

$$\delta L = E_3(L(q,\dot q,\ddot q,\dddot q))\delta q + \frac{dC}{dt} \tag{25}$$

where we now have, by virtue of the extremization of the action, and the quasi-invariance of the Lagrangian, respectively, the equations of motion and the Noether charge of the boundary term of the action for the Lagrangian:

$$E_3(L(q,\dot q,\ddot q,\dddot q)) = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot q} - \frac{d^3}{dt^3}\frac{\partial L}{\partial \dddot q} = 0 \tag{26}$$

$$C = E_2(L(\dot q,\ddot q))\delta q + E_1(L(\ddot q,\dddot q))\delta\dot q + E_0(L(\dddot q))\delta\ddot q \tag{27}$$

and where the Noether charge for the third-order Lagrangian C can also be rewritten as

$$C = \left(\frac{\partial L}{\partial \dot q} - \frac{d}{dt}\frac{\partial L}{\partial \ddot q} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \dddot q}\right)\delta q + \left(\frac{\partial L}{\partial \ddot q} - \frac{d}{dt}\frac{\partial L}{\partial \dddot q}\right)\delta\dot q + \frac{\partial L}{\partial \dddot q}\delta\ddot q \tag{28}$$

2.4 *nth* order lagrangians, $L = L(q, Dq, D^2q, \dots, D^nq)$

Applying induction, we can obtain the Euler-Lagrange equations of nth order (order n), just be careful notation of the derivatives. $D = d/dt$, $D^2 = d^2/dt^2$, ..., $D^n = d^n/dt^n$ are the time derivatives of order 1 to order n (generally $D^0f = 1f = f$. It is not difficult to deduce the following inductively:

$$\delta L = E_n(L(q, Dq, \dots, D^nq))\delta q + \frac{dC}{dt} \tag{29}$$

where we now have the Euler-Lagrange equations and the Noether charge

$$E_n(L) = \sum_{j=0}^{n} D^j \frac{\partial L}{\partial D^j q} = 0 \tag{30}$$

$$C = \sum_{k=0}^{n} E_k \delta D^{n-k-1} q - \Lambda \delta q \tag{31}$$

either

$$C = \left(\frac{\partial L}{\partial Dq} - D\frac{\partial L}{\partial D^2q} + D^2\frac{\partial L}{\partial D^3q} - \cdots - \Lambda\right)\delta q +$$
$$+ \left(\frac{\partial L}{\partial D^2q} - D\frac{\partial L}{\partial D^3q} + \cdots\right)\delta Dq + \cdots + \frac{\partial L}{\partial D^nq}\delta D^{n-1}q$$

3 Fields and lagrangian densities

The previous case can be generalized when we pass from discrete coordinates of particles $q(t)$ to fields in a certain manifold (in physics, generally space-time, although this is not always the only manifold studied). A field $\phi(x) = \phi(x^\mu)$ depends on the coordinates x^μ of the space-time manifold. The Lagrangian L becomes a Lagrangian density $\mathcal{L}$, and the action is defined analogously on that manifold

$$S = \int_M \mathcal{L} = \int_M \mathcal{L}(\phi, \partial\phi, \partial^2\phi, \dots, \partial^s\phi) d^D x \tag{32}$$

where D is now the dimension (not to be confused with the D used for the time derivative earlier). The time derivative is now just one of the partial derivatives $\partial = \partial_\mu$. Mutatis mutandis, if we change $q(t)$ to $\phi(x)$, time derivatives to partial derivatives, all of the above can be generalized. Now the charge is a current J (the Noether charge would essentially be the volume integral of the time component of the current) which will be conserved. This is because the continuity equation has the form

$$\partial_\mu J^\mu = 0 = \partial_0 J^0 + \partial_i J^i = \partial_t Q + \partial_i J^i \tag{33}$$

from where

$$Q = -\int \partial_i J^i dt \tag{34}$$

so Q will be conserved when said integral vanishes. The quasi-invariance condition of the Lagrangian now becomes a quasi-invariance of a divergence, that is, $\mathcal{L}$ can change as a gauge transformation to a partial derivative of a quantity λ^μ holding the integral action invariant, since $\partial_\mu \lambda^\mu$ on the edges or border will not generally contribute to the classical equations of motion.

3.1 First order lagrangian densities

Equations of motion:

$$E_1(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 \tag{35}$$

Noether current:

$$J^\mu = \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi - \lambda^\mu \right) \tag{36}$$

3.2 Second order lagrangian densities

Equations of motion:

$$E_2(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} = 0 \tag{37}$$

Noether current:

$$J^\mu = \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} - \lambda^\mu \right) \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \delta \partial_\nu \phi \tag{38}$$

3.3 Third order lagrangian densities

To simplify the notation, we will indicate the successive derivatives by indices. Thus, $\partial_\mu \partial_\nu \phi = \phi_{\mu\nu}$, ... $\partial^s \phi = \partial_{\mu_1} \cdots \partial_{\mu_s} \phi = \phi_{\mu_1 \cdots \mu_s}$. With this notation, we write the equations of motion and the Noether current as follows.

Equations of motion:

$$E_3(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_\mu} + \partial_{\mu\nu} \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} - \partial_{\mu_1 \mu_2 \mu_3} \frac{\partial \mathcal{L}}{\partial \phi_{\mu_1 \mu_2 \mu_3}} = 0 \tag{39}$$

Noether current:

$$\begin{aligned} J^\sigma = & \left(\frac{\partial \mathcal{L}}{\partial \phi_\sigma} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{\mu\sigma}} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu\sigma}} - \lambda^\sigma \right) \delta\phi + \\ & + \left(\frac{\partial \mathcal{L}}{\partial \phi_{\sigma\nu}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu\sigma}} \right) \delta\phi_\nu + \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu\sigma}} \delta\phi_{\mu\nu} \end{aligned}$$

3.4 nth order lagrangian densities

Equations of motion:

$$\delta \mathcal{L} = E_n(\mathcal{L}(\phi, D\phi, \ldots, D^n \phi))\delta\phi + \partial_\mu \lambda^\mu \tag{40}$$

and now the Euler-Lagrange equations for the field and the Noether current take the functional forms

$$E_n(\mathcal{L}) = \sum_{j=0}^{n} D^j \frac{\partial \mathcal{L}}{\partial D^j \phi} = 0 \tag{41}$$

$$J = \sum_{k=0}^{n} E_k \delta D^{n-k-1} \phi - \lambda \delta\phi \tag{42}$$

equivalently

$$J = \left(\frac{\partial \mathcal{L}}{\partial D\phi} - D\frac{\partial \mathcal{L}}{\partial D^2\phi} + D^2\frac{\partial \mathcal{L}}{\partial D^3\phi} - \cdots - \lambda \right) \delta\phi +$$
$$+ \left(\frac{\partial \mathcal{L}}{\partial D^2\phi} - D\frac{\partial \mathcal{L}}{\partial D^3\phi} + \cdots \right) \delta D\phi + \cdots + \frac{\partial \mathcal{L}}{\partial D^n\phi}\delta D^{n-1}\phi$$

4 The 2 theorems

4.1 Mathematical background

Let n be fields $\phi^i(x)$, $i = 1, \ldots, n$, which depend on D variables or coordinates $x = x^\mu = (x^1, x^2, \ldots, x^D)$. The first-order Euler-Lagrange equations (the discussion can be extended to any order of derivatives by using jet bundles) are written

$$E_i(\varphi) = \frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} = 0 \tag{43}$$

For the action integral

$$S = \int_M \mathcal{L}(x; \phi^i, \partial_\mu \phi^i) d^D x \tag{44}$$

coordinate transformations and fields are defined:

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu \tag{45}$$

$$\phi^i(x) \to \phi'^i(x') = \phi^i(x) + \delta\phi^i = \phi^i(x) + \bar{\delta}\phi^i + \partial_\mu \phi^i \delta x^\mu \tag{46}$$

which are infinitesimal transformations of the quantities x^μ, ϕ^i. At the lowest order, the action changes

$$\delta S = \int_M \mathcal{L}(x', \phi'(x'), \partial\phi'(x')) d^D x' - \int_M \mathcal{L}(x, \phi(x), \partial\phi) d^D x \tag{47}$$

to provide

$$\delta S = \int_M \left[E_i(\phi)\bar{\delta}\phi^i + \partial_\mu B^\mu(x, \phi, \partial\phi, \delta x, \delta\phi) \right] \tag{48}$$

and where B^μ, $\mu = 1, 2, \ldots, D$ are linear functions on $\delta x^\mu, \delta\phi^i$. With these expressions, Noether stated her two theorems.

4.2 Theorem 1 (Noether, r-parametric groups)

The link between symmetries (invariances) and conservation laws is the content of Noether's first theorem. The importance of Noether's theorem resides in the connection of symmetries (or laws of invariance) with the conservation laws. Thus, translation invariance of the Lagrangian implies conservation of momentum, time translation invariance implies conservation of energy (more generally, spatiotemporal translation invariance implies conservation of the so-called energy-momentum-momentum tensor), rotation invariance implies conservation of angular momentum (usually a bivector, but dual to a vector in 3d), boost invariance implies that the center of mass (or center of energy in relativistic version) moves with uniform motion ,...And more generally, Noether's first theorem indicates that the invariance of the action (quasi-invariance of the Lagrangian) against a certain group of transformations (fixing the type of group at a mathematical level is important, generally continuous groups are preferred, usually Lie-Backlünd, but they can be more esoteric groups such as the Poincaré group, the conformal group, the gr de Sitter type, and several others).

Thus, Noether's first theorem establishes a one-to-one correspondence (Noether proved this) and the converse, that a conservation law is associated with a symmetry or invariance, at the same time that a symmetry or invariance is associated with a conserved quantity. even if it is not trivial. The conservation laws of continuous groups are additive laws, while discrete symmetries obey multiplicative conservation laws (for example, invariance under time reversal, invariance under the change of particles by antiparticles, or invariance under specular reflection with the so-called symmetries T, C and P, and are also symmetries in Physics in a combined way, but apparently not separately).

In short, Noether's first theorem states that conservation laws imply invariance and vice versa. If the action integral is invariant for an r-parametric Lie group (Lie-Backlünd today):

$$x^\mu \to x'^\mu = f^\mu(x, \varphi; \varepsilon^1, \ldots, \varepsilon^r) \tag{49}$$

$$\phi^i(x) \to \phi'^i(x') = F^i(x, \varphi; \varepsilon^1, \ldots, \varepsilon^r) \tag{50}$$

where the values

$$\varepsilon = \varepsilon^\rho = 0 = (0, \ldots, 0) = (\varepsilon^1, \ldots, \varepsilon^r) \tag{51}$$

with $\rho = 1, \ldots, r$ give the identity transformation. So, for the parameterization

$$\delta x^\mu = X^\mu{}_\rho(x, \phi)\varepsilon^\rho, \quad |\varepsilon| << 1 \tag{52}$$

$$\delta\phi^i = Z^i{}_\rho(x, \phi)\varepsilon^\rho \tag{53}$$

it is shown, as Noether did, that there exist r conserved currents

$$J^\mu{}_\rho = T^\mu{}_\nu X^\nu{}_\rho - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} Z^i{}_\rho, \quad \rho = 1, 2, \ldots, r \tag{54}$$

with the energy-momentum-impulse tensor given by

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \partial_\nu \phi^i - \delta^\mu{}_\nu \mathcal{L} \tag{55}$$

114 / 5.000 Resultados de traducción Resultado de traducción for the solutions $\phi^i(x)$ of the equations of motion

$$E_i(\phi) = 0$$

of a first-order Lagrangian.

Example 1. Space-time translations. Coordinate transformations

$$x^\mu \to x^\mu + \varepsilon^\mu \delta\phi^i = 0 \tag{56}$$

produce the currents

$$J^\mu{}_\nu = T^\mu{}_\nu, \quad \mu, \nu = 1, 2, 3, \ldots, D \tag{57}$$

Example 2. Internal symmetries. Let be the internal transformations of the fields

$$\delta x^\mu = 0 \tag{58}$$

$$\phi'^i(x) = Y^i_p(\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^r)\phi^p(x) \tag{59}$$

Then the following currents are conserved assuming valid equations of motion $E_i(\phi) = 0$.

$$J^\mu{}_\rho = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu(\phi^i)}Z^i{}_\rho, \quad \rho = 1, 2, \ldots, r \tag{60}$$

The converse also holds: if there are ρ conserved quantities, then there are groups of transformations that leave the action invariant.

4.3 Theorem 2 (Noether, ∞-parametric groups or gauge transformations)

Noether's second theorem connects differential identities (dependencies of the equations of motion) with gauge invariance. Noether's second theorem is much more abstract and subtle, and generally does not have appreciable consequences like the first. If the invariance group is replaced by an infinite-dimensional group and the transformations are gauges of a given set of functions up to a given order in the derivatives, Noether proved that not all equations of motion are independent and there is redundancy or dependency between them. . The set of these relations, generally in the form of identities in the form of differential operators, is called Bianchi identities (although this term is often confused with the Bianchi identities of certain tensors that may or may not be related to this theorem). However, in modern jargon these relationships between equations of motion in the presence of groups of transformations that depend on arbitrary functions up to a certain order in the derivatives are often called Noether identities.

The mathematical statement of Noether's second theorem is more complicated, and a somewhat archaic version of it is included in the following article(section). The most modern versions use shapes and tensors, plus the fancy jet bundle language of a Lagrangian manifold.

In short, Noether's second theorem shows that invariance under an infinitedimensional group of "gauge" transformations implies functional dependencies (redundancies, differential identities) between the equations of motion and vice versa.

If the action integral is invariant under a gauge group (infinite-dimensional), the elements of the group depend on s-smooth or regular functions $\xi^\rho(x)$, $\rho = 1, \ldots, s$ and their derivatives up to order r_ρ, such that the variations are

$$\overline{\delta}\phi^j = \sum_{\rho=1}^{s} \sum_{\sigma_1,\ldots,\sigma_D=0}^{\sigma_1,\ldots,\sigma_D=r_\rho} (\varepsilon^j(x,\phi;\partial\phi)_{\rho,\sigma_1,\ldots,\sigma_D} \frac{\partial^{\sigma_1+\cdots+\sigma_D}}{\partial(x^1)^{\sigma_1}\cdots\partial(x^D)^{\sigma_D}}\xi^\rho(x) \tag{61}$$

and there exist s-identities called Noether identities (or Bianchi identities) given by the expressions

$$\sum_{\sigma_1,\ldots,\sigma_D=0}^{\sigma_1+\cdots+\sigma_D=r_\rho}(-1)^{\sigma_1+\cdots+\sigma_D}\frac{\partial^{\sigma_1+\cdots+\sigma_D}}{\partial(x^1)^{\sigma_1}\cdots\partial(x^D)^{\sigma_D}}\left(\varepsilon(x,\phi,\partial\phi)_{\rho,\sigma_1,\ldots,\sigma_D}E_i(\phi)\right)=0 \tag{62}$$

where $\rho = 1,\ldots,s$, and which give relationships or dependencies between the n-Euler-Lagrange equations. The proof uses integration and the fact that one can choose $\xi^\rho = 0$ and also that we can write

$$\frac{\partial^{\sigma_1+\cdots+\sigma_D}}{\partial(x^1)^{\sigma_1}\cdots\partial(x^D)^{\sigma_D}}\xi^\rho = 0 \tag{63}$$

on the border or edge of M, topologically denoted by ∂M.

Example 1. Classical electrodynamics. In classical electrodynamics one has to

$$E_\mu(A) = \partial^\nu F_{\nu\mu} \tag{64}$$

y

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu \tag{65}$$

The invariant action reads

$$S(A,F) = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \tag{66}$$

Under gauge transformations

$$\delta A^\mu = \partial^\mu \xi(x) \tag{67}$$

we deduce

$$\partial^\mu E_\mu(A) = 0 \tag{68}$$

which is an obvious result from the antisymmetry of F, $F_{\mu\nu} = -F_{\mu\nu}$.

Example 2. General Relativity. In General Relativity (RG), we have the nonlinear field equations

$$E_{\mu\nu}(g) = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = G_{\mu\nu} = \kappa T_{\mu\nu} \tag{69}$$

and where $R_{\mu\nu}$ is the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar of curvature. The invariance of the Einstein-Hilbert integral action

$$S_{EH} = \int d^4x\sqrt{-g}R \tag{70}$$

under diffeomorphisms or general coordinate transformations parameterized in the form

$$\delta x^\mu = \Xi^\mu(x) \tag{71}$$

$$\delta g_{\mu\nu} = D_\mu \Xi_\nu + D_\nu \Xi_\mu \tag{72}$$

where D_μ is the covariant derivative, gives 4 Bianchi identities (Noether identities, dependencies) given by

$$D^\mu E_{\mu\nu}(g) = 0 \tag{73}$$

which is obvious from the symmetry of $G_{\mu\nu}$ and expected from the conservation of the energy-momentum-momentum tensor $T_{\mu\nu}$. These relationships were discovered by Hilbert and discussed by him in his first paper on general relativity in 1915.

5 Algebra and invariant theory

Emmy Noether(1882-1935) earned her doctorate (Ph.D) in Mathematics in 1907 at the University of Erlangen. Another of Noether's facets was the theory of algebraic invariants. The theory of algebraic invariants studies multilinear forms with the formal expression

$$F^{D,p}(y_1, \ldots, y_n; \varepsilon) = \sum_{i_1,\ldots,i_D=1}^{n} a_{i_1 i_2 \cdots i_D} (y_{i_1})^{\alpha_{i_1}} \cdots (y_{i_D})^{\alpha_{i_D}} \tag{74}$$

where $\alpha_{i_1} + \cdots + \alpha_{i_D} = p$. If one passes from variables y_j to variables x_i by linear transformations $x_i = C_{ij} y_j$, $|C_{ij}| \neq 0$, and we insert these relations in the preceding formulae, we hope to find a new form of the same type

$$G^{D,p}(x; b) = F^{D,p}\left[y(x; c); a\right] \tag{75}$$

where the coefficients $b^{j_1 \cdots j_D}$ are functions of the coefficients $a^{j_1 \cdots j_D}$ through the matrix of elements C_{ij}. The main question of the theory of algebraic invariants is: which algebraic functions $f(a)$ of the coefficients $a^{j_1 \cdots j_D}$ are invariant under linear transformations, such that the relation

$$f(b) = |C_{ij}|^g f(a) \tag{76}$$

where g is a rational number? It turns out that there is a deep connection between algebraic invariants and differential invariants, which was discovered as early as the 19th century. Riemann's own geometry, essential in the General Theory of Relativity, uses differential invariants of the type

$$f(x, dx) = g_{ij}(x) dx^i dx^j \tag{77}$$

although more complicated invariants (other geometries) could also be studied in principle, of the type

$$f(x, dx) = g_{i_1 i_2 \cdots i_p} dx^{i_1} \cdots dx^{i_p} \tag{78}$$

which corresponds to the so-called Finsler geometries.

Noether's theorems are relevant in the study of the movement of central forces of the inverse square type (Kepler's problem, with non-trivial hidden symmetry), the study of asymmetric tops in more dimensions. For example, it can be generalized to the denominated group $SO(n)$ the usual group $SO(3)$ of a body or rigid top with Euler equations (do not confuse these equations with the Euler-Lagrangre equations):

$$\frac{dJ_i^B}{dt} + \sum_{j,k=1}^{n} C_{ijk} \omega^j J^{kB} = M_i^B \tag{79}$$

with $i = 1, 2, \ldots, n$. In addition, it also has application in (super)string theories, Quantum Mechanics, Quantum Field Theories (QFT), solid state theories, even black hole theory (the Kerr black hole with mass and angular momentum has a constant of non-trivial motion called Carter's constant associated with a non-trivial symmetry of the black hole, called Killing symmetry), Cosmology, or in the more human Science of the atmosphere in fluids (where quantities such as vorticity and enstrophy play an important role in the nonlinear dynamics associated with them).

Noether's theorems are so beautiful because of their generality and the relevance of symmetry in the world and in the Universe. Although they can be formulated today with a greater elegance, generalization, abstraction and level of sophistication (perhaps I will write a third party on this aspect), the essence remains the same:

- Continuous symmetries imply conservation laws, and vice versa. Mathematically, it is expressed as continuity equations (fields) or quantities whose time derivative is zero (in the case of particle systems).
- Gauge symmetries imply dependencies or functional relationships between equations of motion of fields (particles), and vice versa. Mathematically, it is expressed as identities between certain differential operators in ordinary type equations (particle systems) or in partial derivatives (field theories).

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A Emmy Noether

Figure 1: Emmy Noether portrait in her youth.

B Invariant variation problems

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2. 17

Figure 2: The first page of [6], a legendary paper and cult among theoretical physicists.