Electromagnetic duality: a brief vademecum

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Abstract

A short introduction to duality in (3+1)d via Maxwell equations for undergraduates and advanced graduated is given by these class notes. We demonstrate that fixed the duality mixing angle, we can write always the Maxwell equations in the electric world, and magnetic charges can be hidden and unobservable. The Faraday's law is theoretically derived from the existence of magnetic charges and currents in theoretically dual Maxwell equations. We review the appearance of more general magnetic monopoles in GUT theories, supersymmetry, and gravitational theories. We also introduce the existence of particles with both magnetic and electric charges, called dyons, and the Dirac quantization condition and some generalizations in string theory and M-theory. Finally, we briefly discuss about the experimental searches of magnetic monopoles and dyons, and their cosmological and particle physics relevance. Even when these results are not new, we provide bits of previously unconnected results and formulae more scattered in the literature, from a bottom-up approach useful for students and researchers.

1 Introduction

Magnetic monopoles are a theoretical mystery and a deep experimental challenge since Dirac original works, that introduced them into the physics (with a classical-quantum interplay) game by first time in 1931[1]. Even a non reproducible signal was reported by Cabrera in [2] , and it has been pursued since then. Modern experimental searches by [4], and the issue of fundamental particles with both electrical and magnetic charges are tracked into seminal works by [5, 6, 7]. A modern review of experimental magnetic monopoles can be found in [3]. More general works on monopoles in N-form theories and string theory are, e.g., [8, 9, 10]. The cosmological role of monopoles was highlighted by Preskill in[11] and Zeldovich et al. in[13]. Other white papers and bibliographical monopole compendium can be found in[14, 15] and the books[16, 18]. Interesting reports about magnetic monopoles include [12, 17]. For a short introduction to BPS monopoles you can read [19], and for a general introduction

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to magnetism in particle physics [20]. More recently, monopoles and dyons theoretical studies have become important into gravitational theories, not only due to supersymmetry(SUSY) role into the duality revolution via Montonen-Olive duality (e.g.,[24]), but to the have the gravitational analogues in black holes and higher spin fields appear to have deep consequences into the physics [21, 22, 23], where a similar quantization for Dirac gravitipoles with magnetic mass can be found. The role of this mass is yet not fully understood, but pioner works by Zee et al.[25, 26, 27] relate the magnetic mass to energy quantization and transplanckian mass in quatum gravity(if magnetic gravitipoles are superheavy), and, also, the the existence of periodicity in (multiple) temporal dimensions or the existence of closed timelike curves (CTC). These facts can also be associated to vacuum spacetimes in General Relativity(GR) like the Taub-NUT metric[28] (more D-type vacuum solutions include similar results with care), and constraints to magnetic mass is discussed in [29]. The magnetic charges are related to the charges of the graviton and likely the symmetries of the gravitational sector, see e.g. the nice paper by Hull [30], and we can be found similar interested quantization with Chern-Simons couplings, for instance you can see[31], or read the classical papers by [32, 33, 34, 35, 36] where general extended monopoles were introduced for p-form fields. Chern-Simons couplings are closely related to magnetic monopoles in several dimensions, and Chern-Simons gravities as topological theories were originally studied in[37, 38], the relation with SUSY and superstrings can also be found in[39, 40], or even you can read about the relationships between these Chern-Simons terms with Gauss-Bonnet forms in[41]. The Chern-Simons (super)gravities are a full branch of research where the Chilean and Zanelli school provide lots of insights of the relevance of these type of theories, see e.g. the papers and notes in [42, 43, 44, 45, 46]. In particular, models for a non-perturbative definition of M-theory and the original 11D maximal supergravity of Cremmer-Julia-Scherk [52] can be found in[48, 49, 47], and Horava pursed a condensed matter model approximation to M-theory with a 11D CS supergravity model[50]. CS theories with enhanced gauge invariance is discussed in[51].

The structure of this paper is plenty simple: in section 2 we introduce the magnetic charge, current and density, from basic undergraduate differential and matrix calculus. Then, we give a simple proof of the theoretical derivation of the Faraday law of induction from duality in section 3 we also demonstrate with simple calculus that fixed the duality angle, electromagnetic duality is hidden and invisible, i.e., you can also rotate the abstract field space to have no monopole charges in (3+1)d. In section 4 we review briefly the Dirac quantization conditions and the role of monopoles in GUT and cosmology. Finally, in section 5 we review some important remarkable formulae of generalized duality, important in supersymmetry, superstrings, M-theory and gravitational theories of higher spin fields, generally not very well known and scattered in the literature. We conclude with an outlook review section and some open questions, to our knowledge, about electromagnetic duality and its generalizations.

2 Magnetic charge, current and density

If we complete Maxwell's equations in a way that admits magnetic monopoles (point magnetic charges), we must also introduce the magnetic current density $\vec{j}_m = \vec{j}_m(t, x^i)$ and the monopolar density $\rho_m = \rho_m(t, x^i)$. Thus

$$\nabla \cdot \vec{E} = \rho_e/\varepsilon_0 \tag{1}$$

$$\nabla \cdot B = \mu_0 \rho_m \tag{2}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = -\mu_0 \vec{j}_m \tag{3}$$

$$\nabla \times \vec{B} - \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}_e \tag{4}$$

We take the divergence in the third equation

$$\nabla \cdot \left(\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) = \nabla \cdot \left(-\mu_0 \vec{j}_m \right) \tag{5}$$

$$\nabla \cdot (\nabla \times \vec{E}) + \frac{\partial \nabla \cdot \vec{B}}{\partial t} = -\mu_0 \nabla \cdot \vec{j}_m \tag{6}$$

$$\nabla \cdot (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} \nabla \cdot \vec{B} - \mu_0 \nabla \cdot \vec{j}_m \tag{7}$$

$$0 = -\frac{\partial}{\partial t} \nabla \cdot \vec{B} - \mu_0 \nabla \cdot \vec{j}_m \tag{8}$$

ya que la divergencia de un rotacional es cero (de la condición cohomológica $d^2 = 0$). Usando la ecuación segunda de Maxwell (Ley de Gauss magnética con monopolo), tenemos

$$\nabla \cdot \vec{j}_m + \frac{\partial \rho_m}{\partial t} = 0$$

since the divergence of a curl is zero (from the cohomological condition $d^2 = 0$). Using Maxwell's second equation (magnetic Gauss law with monopole), we have

$$\nabla \cdot \vec{j}_e + \frac{\partial \rho_e}{\partial t} = 0$$

By simply changing the magnetic label for the electric one (or vice versa), you get one from the other. It is the essence of the so-called *electromagnetic duality*. In fact the equations (1), (2),(3),(4) are invariant under the following duality transformations:

$$\vec{E} \to \vec{B} \tag{9}$$

$$c\vec{B} \to -\vec{E} \tag{10}$$

$$c\vec{\rho_e} \to \rho_m \tag{11}$$

$$\rho_m \to -c\rho_e \tag{12}$$

$$c\vec{j}_e \to \vec{j}_m \tag{13}$$

$$\vec{j}_m \to -c\vec{j}_e \tag{14}$$

In fact, these discrete transformations are nothing more than a particular case of a more general set of electromagnetic duality transformations, which rotate electric and magnetic charges, densities, and their respective currents[1]:

$$\star\vec{E} = \vec{E}\cos\theta + c\vec{B}\sin\theta \tag{15}$$

$$c\star\vec{B} = -\vec{E}\sin\theta + c\vec{B}\cos\theta \tag{16}$$

$$c\star\rho_e = c\rho_e\cos\theta + \rho_m\sin\theta \tag{17}$$

$$\star\rho_m = -c\rho_e\sin\theta + \rho_m\cos\theta \tag{18}$$

$$c\star\vec{j}_e = c\vec{j}_e\cos\theta + \vec{j}_m\sin\theta \tag{19}$$

$$\star\vec{j}_m = -c\vec{j}_e\sin\theta + \vec{j}_m\cos\theta \tag{20}$$

or in matrix format

$$\star\begin{pmatrix}\vec{E}\\ c\vec{B}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\vec{E}\\ c\vec{B}\end{pmatrix} \tag{21}$$

$$\star\begin{pmatrix}c\rho_e\\ \rho_m\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}c\rho_e\\ \rho_m\end{pmatrix} \tag{22}$$

$$\star\begin{pmatrix}c\vec{j}_e\\ \vec{j}_m\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}c\vec{j}_e\\ \vec{j}_m\end{pmatrix} \tag{23}$$

Discrete duality transformations are nothing more than the case $\theta = \pi/2$ in the equations (21)-(23)(or in (15)-(20)). Note that it is a curious symmetry that relates vectors, $\vec{E}, \vec{j}_e$, with pseudovectors $\vec{B}, \vec{j}_m$, while ρ_e is a scalar, and in some way ρ_m, θ are pseudoscalars. The angle θ parameterizes the duality and is a kind of rotation in the field space, and is usually called the mixing angle of the real two-dimensional abstract charge space or the angle of duality (in a complex way the group $SO(2)$ can be related with $U(1)$, as is well known in group theory).

3 Faraday's law as conservation of magnetic charge

Suppose we postulate the indestructibility of magnetic charge as follows: "Isolated magnetic charge exists somewhere in the universe, and is indestructible in the same way that electric charge exists and is indestructible, by means of a continuity equation."

The continuity equations of the previous section, and Maxwell's equations symmetrized through duality "Dirac", allow us to derive Faraday's law of induction, a priori a completely empirical law and without any theoretical foundation. Assuming the existence of magnetic charges, a Coulomb law for magnetic fields

[1]More generally, in the language of differential forms , is generalized using the Hodge star operator.

is suggested:

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \rho_m(x')\frac{\vec{x}-\vec{x}' d^3x'}{|\vec{x}-\vec{x}'|^3} = -\frac{\mu_0}{4\pi}\int_V \rho_m(x')\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x'$$

If magnetic currents exist, there will also be a Biot-Savart law for electric fields by symmetry:

$$\vec{E} = -\frac{\mu_0}{4\pi}\int_V \vec{j}_m(x') \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} d^3x' = \frac{\mu_0}{4\pi}\int_V \vec{j}_m(x') \times \nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x'$$

Taking the rotational of this last expression, and applying the identity

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla\vec{A}) - \nabla^2\vec{A}$$

we get

$$\nabla \times \vec{E} = \frac{\mu_0}{4\pi}\int_V \nabla \times (\vec{j}_m(x') \times \nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x' = \tag{24}$$

$$= \frac{\mu_0}{4\pi}\int_V \vec{j}_m(x')\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x' - \frac{\mu_0}{4\pi}\int_V \left[\vec{j}_m(x')\cdot\nabla'\right]\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x' \tag{25}$$

Assuming this to be valid, using the fact that magnetic currents can also vary with time, and the representation of the delta function

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\delta(\vec{x}-\vec{x}')$$

we can use the generation of the magnetic current for time-varying magnetic charge distributions, and the continuity equation for the magnetic charge. Formally:

$$\nabla \times \vec{E} = -\mu_0\int \vec{j}_m(\vec{x}',t)\delta(\vec{x}-\vec{x}')d^3x - \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int_V \rho_m(t,\vec{x}')\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x' \tag{26}$$

and

$$\nabla \times \vec{E} = -\mu_0\vec{j}_m - \frac{\partial\vec{B}(\vec{x},t)}{\partial t} \tag{27}$$

which is one of our Maxwell equations with electromagnetic duality. Finally, we look for a rotation of duality that takes us to the electrical world of the charge space, where $\vec{j}_m = \vec{0}$, that is, we look for a rotation of duality such that

$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}c\vec{j}_e\\ \vec{j}_m\end{pmatrix} \to \begin{pmatrix}c\vec{j}_e\\ \vec{0}\end{pmatrix} \leftrightarrow AJ = J' \tag{28}$$

Inverting the transformation

$$\begin{pmatrix}c\vec{j}_e\\ \vec{j}_m\end{pmatrix} \to \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}c\vec{j}_e\\ \vec{0}\end{pmatrix} \to \begin{pmatrix}c\vec{j}_e\\ \vec{j}_m\end{pmatrix} \to \begin{pmatrix}\cos\theta c\vec{j}_e\\ \sin\theta\vec{j}_e\end{pmatrix} \tag{29}$$

These equations imply

$$c\vec{j}_e = \cos\theta c\vec{j}_e \tag{30}$$

$$\vec{j}_m = \sin\theta\vec{j}_e \tag{31}$$

or else

$$\frac{c\vec{j}_e}{\vec{j}_m} = \frac{1}{\tan\theta}c \rightarrow \vec{j}_m = \tan\theta\vec{j}_e \tag{32}$$

or, writing $c\vec{j}_e \rightarrow \vec{j}_e = \vec{j}'_e$,

$$\vec{j}_m = c\tan\theta\vec{j}_e \tag{33}$$

All this means that for a fixed value of θ, not necessarily a particular one but any fixed value, we can rewrite Maxwell's equations so that they are always written in the "electric world", without magnetic charges (monopoles) nor magnetic currents due to them. You simply have to adjust the fixed value (even if it is arbitrary) and we can always rewrite Maxwell's equations without monopoles using such abstract rotation. For a mixing angle it is easy to prove that

$$\vec{j}_m = c\vec{j}_e\tan\theta \tag{34}$$

$$\rho_m = c\rho_e\tan\theta \tag{35}$$

Proof.

$$\star\begin{pmatrix} c\rho_e \\ \rho_m \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} c\rho_e \\ \rho_m \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} c\rho_e \\ \rho_e\tan\theta \end{pmatrix} \tag{36}$$

and then

$$\star\begin{pmatrix} c\rho_e \\ \rho_m \end{pmatrix} = \begin{pmatrix} c\rho_e + \sin\theta c\rho_e\tan\theta \\ -\sin\theta c\rho_e + \cos\theta\rho_e\tan\theta \end{pmatrix} = \begin{pmatrix} c\rho_e\cos\theta + \sin\theta c\rho_e\tan\theta \\ 0 \end{pmatrix} \tag{37}$$

where

$$\star\begin{pmatrix} c\rho_e \\ \rho_m \end{pmatrix} = \frac{1}{\cos\theta}\begin{pmatrix} c\rho_e\cos^2\theta + \sin^2\theta\tan\theta \\ 0 \end{pmatrix} = \frac{c\rho_e}{\cos\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{38}$$

By the other hand, for currents

$$\star\begin{pmatrix} c\vec{j}_e \\ \vec{j}_m \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} c\vec{j}_e \\ \vec{j}_m \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} c\vec{j}_e \\ c\vec{j}_e\tan\theta \end{pmatrix} = \tag{39}$$

$$\begin{pmatrix} \cos\theta c\vec{j}_e + \sin\theta c\vec{j}_e\tan\theta \\ -\sin\theta c\vec{j}_e + \cos\theta c\vec{j}_e\tan\theta \end{pmatrix} \tag{40}$$

and then

$$\star\begin{pmatrix} c\vec{j}_e \\ \vec{j}_m \end{pmatrix} = \begin{pmatrix} \cos\theta c\vec{j}_e + \dfrac{\sin^2\theta}{\cos\theta}c\vec{j}_e \\ 0 \end{pmatrix} = \frac{1}{\cos\theta}\begin{pmatrix} \cos^2\theta c\vec{j}_e + \sin^2\theta c\vec{j}_e \\ 0 \end{pmatrix} = \frac{c\vec{j}_e}{\cos\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{41}$$

Q.E.D.
In short, duality, maintaining a constant ratio between electric and magnetic charges in a fixed way, remains as a hidden symmetry since $\rho_m, \vec{j}_m$ do not appear in the transformed equations because they give rise to

$$\star \begin{pmatrix} c\rho_e \\ \rho_m \end{pmatrix} = \frac{c\rho_e}{\cos\theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{42}$$

and

$$\star \begin{pmatrix} c\vec{j}_e \\ \vec{j}_m \end{pmatrix} = \frac{c\vec{j}_e}{\cos\theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{43}$$

which is nothing other than the generalization of duality for charges and currents. In fact, by reversing the duality equations we can write

$$\vec{E} = \star\vec{E}\cos\theta - c\sin\theta \star \vec{B} \tag{44}$$

$$c\vec{B} = \star\vec{E}\sin\theta + \cos\theta \star c\vec{B} \tag{45}$$

$$c\rho_e = \cos\theta c\rho_e - \sin\theta \star \rho_m \tag{46}$$

$$\rho_m = \sin\theta c\rho_e + \cos\theta \star \rho_m \tag{47}$$

$$c\vec{j}_e = \cos\theta c\vec{j}_e - \sin\theta \star \vec{j}_m \tag{48}$$

$$\vec{j}_m = \sin\theta c\vec{j}_e + \cos\theta \star \vec{j}_m \tag{49}$$

or equivalently in matrix form

$$\begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \star\vec{E} \\ \star c\vec{B} \end{pmatrix} \tag{50}$$

$$\begin{pmatrix} c\rho_e \\ \rho_m \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \star c\rho_e \\ \star\rho_m \end{pmatrix} \tag{51}$$

$$\begin{pmatrix} c\vec{j}_e \\ \vec{j}_m \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \star c\vec{j}_e \\ \star\vec{j}_m \end{pmatrix} \tag{52}$$

4 Unobservability of monopoles and dyons

The unobservability of magnetic monopoles and particles with both charges (electric and magnetic), called dyons, implies taking divergence

$$\nabla \cdot \vec{E} = \cos\theta \nabla \times \vec{E} - c\sin\theta \nabla \cdot \star\vec{B} \tag{53}$$

and since

$$\nabla \cdot \star\vec{E} = \star\frac{\rho_e}{\varepsilon_0} = \frac{\rho_e}{\cos\theta\varepsilon_0} \tag{54}$$

$$\nabla \cdot \star\vec{B} = \mu_0 \star \rho_m = \mu_0 \cdot 0 = 0 \tag{55}$$

$$\nabla \cdot \vec{E} = \cos\theta \frac{\rho_e}{\cos\theta\varepsilon_0} = \frac{\rho_e}{\varepsilon_0} \tag{56}$$

$$\nabla \cdot \star\vec{B} = \mu_0 \star \rho_m = \star\mu_0\rho_m = \star\nabla \cdot \vec{B} \tag{57}$$

thus

$$\nabla \cdot \vec{E} = \frac{\rho_e}{\varepsilon_0} \tag{58}$$

$$\nabla \cdot \vec{B} = 0 \tag{59}$$

Similarly, taking the rotational to the fields and their duals

$$\nabla \times \vec{E} = \cos\theta(\nabla \times \star\vec{E}) - c\sin\theta\nabla \times (\star\vec{B}) \tag{60}$$

$$\nabla \times \star\vec{E} = \star\nabla \times \vec{E} = - \star \mu_0\vec{j}_m - \star\frac{\partial\vec{B}}{\partial t} \tag{61}$$

$$\nabla \times \star\vec{B} = \star\nabla \times \vec{B} = \star\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} + \star\mu_0\vec{j}_e \tag{62}$$

or

$$\nabla \times \vec{E} = \mu_0\vec{j}_m - \frac{\partial\vec{B}}{\partial t} \tag{63}$$

$$\nabla \times \vec{B} = \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{j}_e \tag{64}$$

With our choice of general duality transformation, it turns out that $\vec{j}_m = c\tan\theta\vec{j}_e$. Thus, $\star\vec{j}_m = \star c\tan\vec{j}_e$. Therefore:

$$\nabla \times \vec{E} = \cos\theta\left(- \star \mu_0 c\tan\theta\vec{j}_e - \star\frac{\partial\vec{B}}{\partial t}\right) - c\sin\theta\left(\star\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} + \star\mu_0\vec{j}_e\right) = \tag{65}$$

$$= - \star \mu_0 c\sin\theta\vec{j}_e - \star\cos\theta\frac{\partial\vec{B}}{\partial t} - c\sin\theta \star \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} - c\sin\theta \star \mu_0\vec{j}_e = \tag{66}$$

$$= -\frac{\partial\vec{B}}{\partial t} - \mu_0 \star \left(\cos\theta\vec{j}_m + \sin\theta c\vec{j}_e\right) \tag{67}$$

Using Maxwell's equations and dualities

$$\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t} - \mu_0\left[\cos\theta(-c\vec{j}_e\sin\theta) + \cos^2\theta\vec{j}_m + c\sin\theta\cos\theta\vec{j}_e + \sin^2\vec{j}_m\right] \tag{68}$$

or

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \mu_0 \vec{j}_m \tag{69}$$

For our duality transformation we choose $\vec{j}_m = \vec{0}$. Then we have to write

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \tag{70}$$

and

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}_e \tag{71}$$

5 Monopoles and Dirac quantization

Using quantum mechanical arguments added to Maxwell's equations with duality, P. A. M. Dirac deduced [?] that the quantization of the magnetic flux and the quantum phase under duality transformation implies that if we want a well-defined quantum theory, the electric and magnetic charge cannot take any value, but, in Gaussian units

$$\boxed{\frac{Q_e Q_m}{\hbar c} = \frac{n}{2}}$$

or in S.I. units.

$$\boxed{\frac{q_e q_m}{2\pi\hbar} = N}$$

where Q_e, q_e are electric charges, Q_m, q_m are magnetic charges, and N, n are integers. $\hbar = h/2\pi$. We can not stress enough that is quantum mechanics (QM) and the wave-function phase what is behind of these mathematical relationships.

Also, the general strength of a dyon[5, 6] is written

$$\vec{F} = \frac{e_1 g_2 + e_2 g_1}{r^3}\vec{r} + \frac{(e_1 g_2 - e_2 g_1)\frac{\vec{v}}{c} \times r}{r^3} \tag{72}$$

Furthermore, the minimum mass of a monopole can be estimated by the relation

$$M_m = \frac{g_D^2}{e^2}\frac{\varepsilon_0}{\mu_0}\frac{1}{4\alpha_e} m_e \approx 4692 m_e \tag{73}$$

where m_e is the electron mass, and the dual coupling is related to the fine structure constant

$$g_D = \frac{e}{2\alpha_e} = \frac{137e}{2} \tag{74}$$

6 Other electromagnetic dualities

Grand Unified Theories or GUT include superheavy versions of the Dirac monopole, with a mass of the order of $M_m(GUT) > 10^{16}GeV/c^2$[11, 12]. GUT monopoles could be dark matter, although they are problematic in Cosmology, and also catalyze proton decays. You can review these topics more deeply reading[53].

In the case of supersymmetric monopoles, there is the so-called BPS bound[10] that allows estimating a bound lower than its mass theoretically. Furthermore, in supersymmetric theories of strings and branes, the dimensional relationship between branes and their duals exists with the dimension and number of maximum possible supersymmetry, through the following relationship in 10D with gravity and Dp-brane charges:

$$2\kappa_{10}^2 \rho_{Dp}\rho_{D(6-p)} = 2\pi n \tag{75}$$

In M-theory, see e.g.[54], you get a similar striking result relating the electric M2-brane charge, the magnetic dual M5-brane charge and the gravitational constant in 11D with

$$2\kappa_{11}^2 T_{M2} T_{M5} = 2\pi n \tag{76}$$

Generally speaking, matching left-right degrees of freedom in bosonic and fermionic sectors of supersymmetry and supergravity duality theories impose the condition

$$D - d = \frac{1}{2}mn = \frac{MN}{4} \tag{77}$$

Furthermore, also for higher spin fields[21, 22, 23] there is an interesting generalization of duality (extendable to gravitational and electromagnetic dyons):

$$\frac{1}{q!} Q^e_{a_1\cdots a_q} P^{(m)a_1\cdots a_q} = 2\pi\hbar N \tag{78}$$

For $q = s - 1$, with $s = 2$, we obtain

$$\frac{1}{2\pi\hbar} Q_{\gamma_1\cdots\gamma_{s-1}} P^{\gamma_1\cdots\gamma_{s-1}} \in \mathbb{Z} \rightarrow \frac{4GP_\gamma Q^\gamma}{\hbar} \in \mathbb{Z} \tag{79}$$

for

$$\frac{MN}{2\pi\hbar} f_{\gamma_1\cdots\gamma_{s-1}} f^{\gamma_1\cdots\gamma_{s-1}} = n \tag{80}$$

with

$$\Delta\Psi = \frac{N}{\hbar} f_{\gamma_1\cdots\gamma_{s-1}} \int d^3x T^{0\gamma_1\cdots\gamma_{s-1}} = \frac{MN}{2\pi\hbar} f_{\gamma_1\cdots\gamma_{s-1}} f^{\gamma_1\cdots\gamma_{s-1}} \tag{81}$$

Behind these beautiful equations, there are subtle links between mathematics and physics, numbers and functions. Cohomology, differential geometry, supersymmetry algebras, homotopy, and the quantum mechanical wave-function and its observables are entangled with these formulae in such a way that vindicates a further insight.

7 Conclusion

With this brief introduction to electromagnetic duality in (3+1)d, we have seen:

1. Given a fixed θ, the equations can be transformed into the usual Maxwell equations. The angle θ measures how much electric and magnetic charge the particle has.

2. The duality mixing angle measures the fraction of magnetic and electric charge, as well as the fraction of magnetic current and electric current.

3. Particles can have a magnetic and electric charge, being called dyons in this case.

4. Electromagnetic duality is a theoretical tool that allows explaining a phenomenological law such as the Faraday-Lenz law.

5. The existence of a single magnetic monopole in the universe implies the quantization of electric charge[1], which is a verified empirical fact (every charged particle is an integer multiple of a fundamental quantity).

$$\frac{eg}{\hbar c} = \frac{n}{2}$$

6. There is a god condition due to Julian Schwinger.

$$\left(\frac{e_1 g_2 - e_2 g_1}{2\pi\hbar c}\right) = n$$

7. There is no experimental evidence yet for the existence of magnetic monopoles or dyons[2].

8. GUT theories naturally include superheavy versions of magnetic monopoles, called GUT monopoles. They can catalyze proton decay or could even be candidates for dark matter, but in general they pose a cosmological problem.

9. Electromagnetic duality can be extended to the gravitational sector, although it is less known, including high spin fields.

$$\frac{MN}{2\pi\hbar} f_{\gamma_1\cdots\gamma_{s-1}} f^{\gamma_1\cdots\gamma_{s-1}} = \frac{Q_E^{(e)} Q^{(m)M}}{2\pi\hbar} = n$$

   where E, M are spin multiindexes $(E, M) = a_1, \cdots, a_q = \gamma_1, \cdots, \gamma_{s-1}$.

10. We can have gravitational dyons too, and they could be both, subplanckian or transplanckian mass particles.

We can list some known (to our knowledge) unanswered questions about monopoles, dyons, generalized dyons and electromagnetic (and gravitational) duality:

1. Do magnetic monopoles (point-like or extended) exist physically in our Universe? If so, what are their types, symmetry groups and masses?

2. What are the full symmetry origins of the whole duality groups and their gravitational analogues?

3. Do physically existing monopoles catalyze proton decays? Is baryon number conserved in proton decays catalyzed but physical monopoles?

4. Do magnetic masses exist? Are they subplanckian or transplanckian?

5. Are electromagnetic dyons and charges or their gravitational analogues present in black hole atmospheres?

6. Were magnetic monopoles produced in early stages and the Universe diluted by inflation or some other mechanism acted on them?

7. What phase transition made monopoles to get diluted in the early Universe?

8. Is really the existence of magnetic monopoles the reason of electric charge quantization?

9. If no inflation hints were found[2] What would solve the monopole problem in Cosmology? By the contrary, were inflation or monopole discovered, what theoretical consequences and models would be coherent with the mass spectrum of monopoles or the tensor-to-scalar ratio metric perturbations found? What to expect from a tiny r-parameter from inflation or monopoles?

10. What are the magnetic monopole contribution to the energy-density of the Universe?What about other topological defects like cosmic strings, domain walls,...?

11. What are the strength of GW signals and gravitational effects of magnetic monopoles in current and future gravitational or particle physics detectors?

12. Could we detect magnetic monopoles with next generation DM detectors in the near future?

There are likely lots of additional questions beyond the above list, but our intentions are humble and only review some of the open questions about experimental and theoretical researches involving magnetic monopoles and their generalizations. We inquiry eager readers to consult our given bibliography and references therein to further study and research these and other topics.

[2]OK, I know...Everyone supposes inflation does exist, but what if not? What dilutes the copious production of magnetic monopoles if no evidence of inflation is got in future experiments and observations?

References

[1] Dirac, Paul (1931). *Quantised Singularities in the Electromagnetic Field.* Proceedings of the Royal Society A. 133 (821). https://doi.org/10.1098%2Frspa.1931.0130

[2] Cabrera, Blas (May 17, 1982). *First Results from a Superconductive Detector for Moving Magnetic Monopoles.* Physical Review Letters. 48 (20): 1378–1381. https://doi.org/10.1103%2FPhysRevLett.48.1378

[3] Arttu Rajantie (2016). *The search for magnetic monopoles.* Physics Today. 69 (10): 40. https://doi.org/10.1063%2FPT.3.3328

[4] Aad, Georges et al (2020). *Search for magnetic monopoles and stable high-electric-charge objects in 13 TeV proton-proton collisions with the ATLAS detector.* Phys. Rev. Lett. 124 (3): 031802. https://arxiv.org/abs/1905.10130,https://doi.org/10.1103%2FPhysRevLett.124.031802

[5] Julian Schwinger, *Magnetic charge and the charge quantization condition.* Phys. Rev. D 12, 3105 – Published 15 November 1975. https://journals.aps.org/prd/abstract/10.1103/PhysRevD.12.3105

[6] J. S. Schwinger, *Sources and magnetic charge*, Phys. Rev. 173 (1968) 1536.

[7] J. S. Schwinger, *A magnetic model of matter*, Science 165 (1969) 757.

[8] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, *Duality, self-duality, sources and charge quantization in abelian N-form theories*, Phys. Lett. B 400 (1997) 80.Arxiv:https://arxiv.org/abs/hep-th/9702184

[9] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, *p-Brane dyons and electric-magnetic duality*, Nucl. Phys. B 520 (1998) 179.https://arxiv.org/pdf/hep-th/9712189.DOI:https://doi.org/10.1016%2FS0550-3213%2898%2900179-5

[10] Polchinski, Joseph (February 1, 2004). *Monopoles, Duality, and String Theory.* International Journal of Modern Physics A. 19 (supp01): 145–154. https://arxiv.org/abs/hep-th/0304042. https://doi.org/10.1142%2FS0217751X0401866X

[11] Preskill, John (1979). *Cosmological production of superheavy magnetic monopoles.* Phys. Rev. Lett. 43 (19): 1365–1368. https://doi.org/10.1103%2FPhysRevLett.43.1365

[12] Preskill, John (1984). *Magnetic Monopoles.* Annu. Rev. Nucl. Part. Sci. 34 (1): 461–530. https://doi.org/10.1146%2Fannurev.ns.34.120184.002333

[13] Zel'dovich, Ya. B.; Khlopov, M. Yu. (1978). *On the concentration of relic monopoles in the universe*. Phys. Lett. B79 (3): 239–41. https://doi.org/10.1016%2F0370-2693%2878%2990232-0

[14] Giacomelli, G. (2000), *Magnetic Monopole Bibliography*, https://arxiv.org/pdf/hep-ex/0005041.pdf

[15] Balestra, S. (2011), *Magnetic Monopole Bibliography-II*, https://arxiv.org/abs/1105.5587

[16] Atiyah, M. F.; Hitchin, N. (1988). *The Geometry and Dynamics of Magnetic Monopoles*. Princeton University Press. ISBN 0-691-08480-7.

[17] Milton, K. A. (2006). *Theoretical and experimental status of magnetic monopoles*. Reports on Progress in Physics. 69 (6): 1637–1711. https://arxiv.org/abs/hep-ex/0602040,https://doi.org/10.1088%2F0034-4885%2F69%2F6%2FR02

[18] Shnir, Y. M. (2005). *Magnetic Monopoles*. Springer. ISBN 978-3-540-25277-1.

[19] Sutcliffe, P. M. (1997). *BPS monopoles*. Int. J. Mod. Phys. A. 12 (26): 4663–4706. https://arxiv.org/abs/hep-th/9707009. https://doi.org/10.1142%2FS0217751X97002504

[20] Vonsovsky, S. V. (1975). *Magnetism of Elementary Particles*. Mir Publishers.

[21] C. Bunster, S. Cnockaert, M. Henneaux, R. Portugues,*Monopoles for Gravitation and for Higher Spin Fields*. ArXiv: https://arxiv.org/abs/hep-th/0601222, DOI:https://doi.org/10.1103/PhysRevD.73.105014, Phys.Rev.D73:105014,2006.

[22] Claudio Bunster, Marc Henneaux,*Sources for generalized gauge fields*, ArXiv:http://arxiv.org/abs/1308.2866v2, DOI: https://doi.org/10.1103/PhysRevD.88.085002.

[23] Claudio Bunster, Marc Henneaux, *A Monopole Near a Black Hole*, DOIhttps://doi.org/10.48550/arXiv.hep-th/0703155, ArXiv: https://arxiv.org/abs/hep-th/0703155

[24] *Exact Electromagnetic Duality*, ArXiv: https://arxiv.org/abs/hep-th/9508089.Nucl.Phys.Proc.Suppl. 45A (1996) 88-102; Nucl.Phys.Proc.Suppl. 46 (1996) 1-15.

[25] Zee, A. (1985). *Gravitomagnetic Pole and Mass Quantization*. Physical Review Letters, 55(22), 2379–2381. doi:https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.55.2379. Erratum at ibidem

[26] S. Ramaswamy and A. Sen, *Comment on "Gravitomagnetic pole and mass quantization".*, Phys. Rev. Lett. 57 (1986) 1088. DOI:https://doi.org/10.1103/PhysRevLett.57.1088

[27] Samuel, J.; Iyer, B. R. (1986). *Comment on "Gravitomagnetic Pole and Mass Quantization".* Physical Review Letters, 57(8), 1089–1089. DOI:https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.57.1089

[28] Newman, E.; Tamburino, L.; Unti, T. (1963). *Empty-Space Generalization of the Schwarzschild Metric.* , 4(7), 915–0. DOI:https://doi.org/10.1063/1.1704018

[29] Mueller, M; Perry, M J (1986). *Constraints on magnetic mass.* Classical and Quantum Gravity, 3(1), 65–69. DOI:https://iopscience.iop.org/article/10.1088/0264-9381/3/1/009/pdf

[30] C.M. Hull, *Magnetic Charges for the Graviton*, arXiv:https://arxiv.org/abs/2310.18441. Journal of High Energy Physics, Volume 2024, Issue 05, article id. 257. DOI:https://ui.adsabs.harvard.edu/link_gateway/2024JHEP...05..257H/doi:10.1007/JHEP05(2024)257

[31] Claudio Bunster, Andrés Gomberoff, Marc Henneaux,*Extended Charged Events and Chern-Simons Couplings*, ArXiv:https://ui.adsabs.harvard.edu/link_gateway/2013PhRvD..88h5002B/arxiv:1308.2866, DOI:https://doi.org/10.1103/PhysRevD.84.125012

[32] Claudio Teitelboim (1986). *Monopoles of higher rank.* , 167(1), 69–72. DOI :https://doi.org/10.1016/0370-2693(86)90547-2

[33] Claudio Teitelboim (1986). *Gauge Invariance for Extended Objects* Phys.Lett.B 167 (1986) 63-68. DOI:https://doi.org/10.1016/0370-2693(86)90546-0

[34] Henneaux, Marc; Teitelboim, Claudio (1986). *Quantization of topological mass in the presence of a magnetic pole.* Physical Review Letters, 56(7), 689–692. URL:https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.56.689

[35] Marc Henneaux; Claudio Teitelboim (1986). *p-Form electrodynamics.* , 16(7), 593–617. URL:https://link.springer.com/article/10.1007/BF01889624

[36] Nepomechie, Rafael I. (1985). *Magnetic monopoles from antisymmetric tensor gauge fields.* Physical Review D, 31(8), 1921–1924. DOI:https://doi.org/10.1103/PhysRevD.31.1921

[37] A.H. Chamseddine (1990). *Topological gravity and supergravity in various dimensions.* , 346(1), 0–234. DOI: https://doi.org/10.1016/0550-3213(90)90245-9

[38] , A.H.Chamseddine . *Topological Gauge Theory of Gravity in Five-dimensions and All Odd Dimensions*. Phys.Lett.B 233 (1989) 291-294.

[39] W. Nahm, *Supersymmetries and their representations*. Nuclear Physics B Volume 135, Issue 1, 27 March 1978, Pages 149-166. DOI:https://doi.org/10.1016/0550-3213(78)90218-3.

[40] Michael B. Green (1989). *Super-translations, superstrings and Chern-Simons forms.*, Physics Letters B Volume 223, Issue 2, 8 June 1989, Pages 157-164. DOI.:https://doi.org/10.1016/0370-2693(89)90233-5

[41] Folkert Müller-Hoissen (1990). *From CS to Gauss-Bonnet*. Nucl.Phys.B 346 (1990). (1). 235-252. DOI:https://doi.org/10.1016/0550-3213(90)90246-A

[42] J.Zanelli, *Quantization of the Gravitational Constant in Odd-Dimensional Gravity*, Phys.Rev. D51 (1995) 490-492. ArXiv:https://arxiv.org/abs/hep-th/9406202, DOI:https://doi.org/10.1103/PhysRevD.51.490.

[43] J. Zanelli, *Chern-Simons Forms in Gravitation Theories*. Contribution to:7th Summer School on Geometric, Algebraic and Topological Methods for Quantum Field Theory, 137-188. Lect.Notes Phys. 892 (2015) 289-310.

[44] J. Zanelli,*Introductory lectures on Chern-Simons theories*, AIP Conf.Proc. 1420 (2012) 1, 11-23.DOI:https://doi.org/10.1063/1.3678608

[45] J.Zanelli,*(Super)-Gravities Beyond 4 Dimensions*, Lectures given at the 2001 Summer School "Geometric and Topological Methods for Quantum Field Theory", Villa de Leyva, Colombia, June 2001.https://arxiv.org/abs/hep-th/0206169, DOI:https://doi.org/10.48550/arXiv.hep-th/0206169

[46] J.Zanelli, *Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008)*, ArXiv:https://arxiv.org/abs/hep-th/0502193, DOI:https://doi.org/10.48550/arXiv.hep-th/0502193

[47] Fernando Izaurieta, Eduardo Rodríguez, *On eleven-dimensional Supergravity and Chern-Simons theory*, ArXiv:https://arxiv.org/abs/1103.2182, DOI:https://doi.org/10.1016/j.nuclphysb.2011.10.012

[48] R. Troncoso, J. Zanelli,*New gauge supergravity in seven-dimensions and eleven-dimensions*, Phys. Rev. D 58 (1998) 101703. arXiv:https://arxiv.org/abs/hep-th/9710180, DOI:https://doi.org/10.1103/PhysRevD.58.101703.

[49] R. Troncoso, J. Zanelli, *Gauge supergravities for all odd dimensions*, Int. J. Theor. Phys. 38 (1999) 1181–1206. ArXiv:https://arxiv.org/abs/hep-th/9807029,Int.J.Theor.Phys.38:1181-1206,1999, DOI:https://doi.org/10.1023/A%3A1026614631617.

[50] P. Horava, *M-theory as a holographic field theory*, Phys. Rev. D 59 (1999) 046004. ArXiv:https://arxiv.org/abs/hep-th/9712130, doi:10.1103/PhysRevD.59.046004.

[51] P. Mora,*Chern-Simons branes with enhanced gauge invariance* , Journal of High Energy Physics, Volume 2023, Issue 07, article id. 107.ArXiv:https://arxiv.org/abs/2305.06186

[52] E. Cremmer, B. Julia, and J. Scherk, *Supergravity theory in eleven dimensions*, Phys. Lett. B76, 409-412 (1978).

[53] Maurizio Spurio, *Searches for magnetic monopoles and others stable massive particles.* Arxiv:https://arxiv.org/abs/1906.02039v1,DOI:https://doi.org/10.48550/arXiv.1906.02039

[54] M.J. Duff, *Supermembranes*, ArXiv:https://arxiv.org/abs/hep-th/9611203, DOI:https://doi.org/10.48550/arXiv.hep-th/9611203