HD 226868/CygX-1

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Abstract

Gravitational field assignment.

1 Introduction

You walk into astronomy class one day and find the following question on the board: "What is the mass of Cygnus X-1?" Your class has been learning about binary stars, orbits, and the lifecycle of stars, and you already know that Cygnus X-1 is a well-known X-ray source in our galaxy.

Once everyone is in class, your professor says that the first person to solve this question using astronomical methods and data (instead of looking up the answer on the Internet or in a book) will be excused from homework for the rest of the current unit – a very tempting prospect, since you could use that extra time to sleep!

Your professor tells you that the resources you can use include the University's introductory astronomy equipment (for example, an optical telescope), as well as astronomical data available in print and on-line. Use your understanding of the laws of physics to select an experiment which will help you to find the answer.

2 Calculations with 3 methods

2.1 Methods

You immediately think of three possible approaches:

- Use Newton's law of universal gravitation to solve directly for the mass of Cygnus X-1.
- Use Cygnus X-1's volume and density to deduce its mass.
- Use Kepler's Laws to estimate the mass of Cygnus X-1.

At least one of the above methods will give you the correct answer. Which one do you want to try?

Figure 1: HD 226868/Cygnus X-1 system.

2.2 1st method: universal gravitation

Recap: Your astronomy professor has tasked the class with determining the mass of the black hole candidate, Cygnus X-1. You thought of three possible ways to do this, one of which will give you the right answer. You've decided to try using Newton's law of universal gravitation directly.

You know from the work you've been doing in astronomy class that the mass of Cygnus X-1 could be determined if a probe is launched toward it and programmed to orbit at a fixed distance. The probe would be kept in orbit around Cyg X-1 by the force of gravity, whose strength is dependent only upon Cyg X-1's mass. The probe could then radio back information about its orbital speed, and you would be able to determine the mass of Cyg X-1 with that information.

It would be a simple matter of using Newton's law of universal gravitation and the equations for circular motion.

How would this work mathematically? You realize that if you balance the forces on the probe, you can get the mass. First look at the probe and the forces acting on it. The gravitational attraction of Cygnus X-1 pulls the probe inward, toward the black hole. The centripetal force of the circular motion of the probe pushes it outward, and the two are equally balanced.

So, you take the two forces $F_N = F_c$ such as

$$F_N = G\frac{Mm}{r^2} = m\frac{v^2}{r} = F_c$$

Where $G = G_N$ is the Newton's gravitational constant, M is the mass of Cyg X-1, m is the mass of the probe, r is the distance of the probe from the center (assumed circular for simplicity) of Cyg X-1, and v is the orbital velocity of the probe.

Because m, the mass of the satellite, appears on both sides of the equation, it can be eliminated. Rearranging to solve for M, you get an expression with the measurable values of the probe's speed and distance from Cyg X-1 and the known value of Newton's constant of universal gravitation, G:

$$M = \frac{v^2 r}{G}$$

Remark: Prehistoric man realized a long time ago that when objects are released near the surface of the Earth, they always fall down to the ground. In other words, the Earth attracts objects near its surface to itself.

Galileo (1564-1642) pointed out that heavy and light objects fall toward the earth at the same rate (so long as air resistance is the same for each). But it took Sir Isaac Newton (in 1666) to realize that this force of attraction between masses is universal!

Newton proved that the force that causes, for example, an apple to fall toward the ground is the same force that causes the moon to fall around, or orbit, the Earth. This universal force also acts between the Earth and the Sun, or any other star and its satellites. Each attracts the other.

Sir Isaac Newton defined this attraction mathematically. The force of attraction between two masses is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers. This is all multiplied by a universal constant whose value was determined by Henry Cavendish in 1798. In our local Universe:

$$G_N = G = 6.674 \cdot \frac{Nm^2}{kg^2}$$

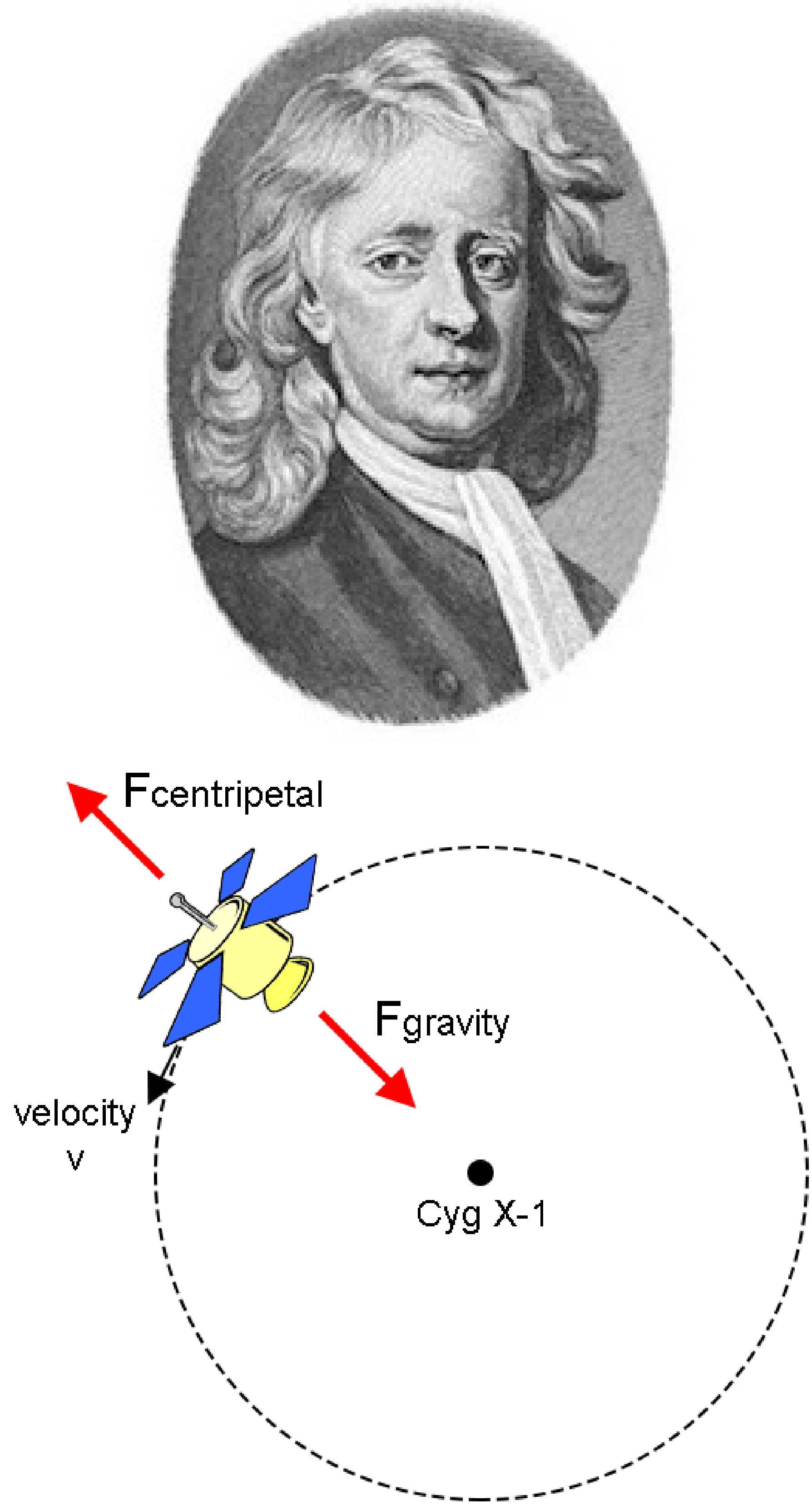

Figure 2: Newton and orbits.

Question 1. The Sun has a mass of $2 \cdot 10^{30}$ kg, and is $1.5 \cdot 10^{8}$ km away, while the mass of the Moon is $7.35 \cdot 10^{22}$ kg, and is $3.85 \cdot 10^{5}$ km away. Which exerts a stronger gravitational influence on the Earth?
☐ The Sun.
☐ The Moon.
☐ They are about the same.
☐ Don't pick me!

Question 2. Cygnus X-1 is orbiting the massive star HDE226868, with a period of 5.6 days. What can you infer from this?
☐ The mass of HDE226868.
☐ The mass of Cygnus X-1.
☐ Nothing.
☐ The mass of the binary system.

Newton's First Law of motion states that an object moving at constant speed will continue that motion unless acted on by an outside force. This means that circular motion can only happen if there is a "center seeking" force – otherwise things would just travel in a straight line, rather than the curved line of a circule. Centripetal means 'center seeking', so centripetal force is used to refer to the force experienced by an object traveling in a circle. For example, when someone spins a ball attached to a rope horizontally above his head, the rope transmits a centripetal force from the muscles of the hand and arm, causing the ball to move in a circular path.

Centripetal forces cause centripetal accelerations. In the special case of the Earth's circular motion around the Sun – or any satellite's circular motion around any celestial body – the centripetal force causing the motion is the result of the gravitational attraction between them.

The arrows (or vectors) show the direction of the circular velocity (v, always tangent to the circular path) and the circular acceleration (a) caused by a centripetal force. Centripetal means center-seeking. Centripetal forces are always directed toward the center of the circular path.

By definition, acceleration is the rate of change in velocity of an object, and velocity is determined by dividing the distance travelled by the time interval it took to cover that distance. In the special case of circular motion, the distance covered is the circumference of a circle or $2\pi r$, where π is the mathematical constant and r is the radius fo the circle. The time interval for an object to travel once around its circular path is called the period and is represented by T.

$$v = \frac{2\pi r}{T}$$

$$F_c = ma_c = m\frac{v^2}{r}$$

Question 3. Knowing what you know about uniform circular motion, and physics, when a NASCAR race car speeds around a circular track at 160 mph what keeps it from flying off the sides of the track?
☐ Gravitational Force.
☐ Centripetal Force.
☐ Frictional Force.
☐ Electromagnetic Force.
☐ Strong Nuclear Force.

Question 4. Imagine you attached a rock to a rope and were swinging it around in a circle very quickly. If you could keep the speed of the rock constant as you pulled the rope in, what would happen to the force transmitted to you by the rope?
☐ Increase inversely with the distance.
☐ Decrease inversely with the distance.
☐ Increase as the square of the distance.
☐ Decrease as the square of the distance.
☐ Remain the Same.

Question 5. Imagine that this time you were aboard the International Space Station while swinging a rock on a rope around in a circle quickly quickly (so you can neglect the effects of gravity) swinging it around in a circle very quickly. Suddenly, the ropa breaks. What did the path of the rock look like before it hit the ISS window a few feet away?
☐ An arc.
☐ A parabola.
☐ A hyperbola.
☐ A line.
☐ It dropped to the ground (and didn't hit the window).

A first step to use our first method to calculate Cyg-X1 mass is to test it with the binary system Earth-Sun.

Because the gravitational attraction of our Sun for the Earth is the centripetal force causing the Earth's circular motion around the Sun, we can use Newton's law of universal gravitation to find the mass of the Sun without visiting the Sun. This is the same technique you would use to determine the mass of Cygnus X-1 with a probe. The Earth, orbiting the Sun, plays the same role as the probe sent to orbit Cygnus X-1.

$$M = \frac{v^2 r}{G}$$

In this equation, we know G, because it is a constant. However, we need to know how far the Earth is from the Sun and how fast it is moving around the Sun.

Question 6. Calculate the earth speed around the sun. Note theta G is known to be $G = 6.674 \cdot 10^{-11} Nm^2/kg^2$ and $r = 150Mkm$. The Earth's velocity around the Sun is just the total distance travelled around a circumference of radius r divided by the time required for the Earth to make one complete orbit around the Sun, T. Then, you can calculate the mass of the sun from these values (with 2 significant figures).

Your idea is to launch a probe to Cyg X-1, let it orbit for a little while, and then report back it's orbital speed in order to used Newton's law of universal gravitation to determine the mass of Cyg X-1. So, let's launch that probe!

First, you decide to take a step back. You know that you need the answer fairly quickly if you want to be the one to answer the challenge question first. Keeping this in mind, you do a quick calculation to see how long this would take.

You know from astronomy class that the speed of light is the absolute speed limit in nature. Your probe would actually travel much slower than that, but if you calculate how long it would take light to get to Cyg X-1 and back again, it will tell you the absolute fastest you could hope to get an answer.

It is an easy matter to calculate this minimum time, once you know the distance to Cyg X-1. The total distance traveled would be twice the distance to Cyg X-1.

$$t = \frac{D}{v} = \frac{2d(CygX1)}{c}$$

Data: A few helpful numbers

- The distance to Cyg X-1 is 2.5 kiloparsecs.
- A kiloparsec is a thousand parsecs.
- A parsec is $3.0857 \cdot 10^{16} m$.
- The speed of light es $3 \cdot 10^{8} m/s$ in our local Universe.

Question 7. Can you calculate Cygnus X-1 mass with these values? Explain your answer.

2.3 2nd method: volume of Cygnus X-1

You learned in science class that density is the amount of matter in a given volume. Density is a characteristic property of matter, which means that density can be used to identify a sample of matter and the size of the sample will not change the result. Density is determined by dividing the mass of a substance by the volume occupied by that mass.

$$D = \frac{M}{V}$$

So, if you know the volume and density of an object, you can find the mass by simply rearranging the above equation.

$$M = DV$$

The image was taken by the optical Schmidt telescope at the Palomar Observatory and shows the area of sky occupied by the black hole Cygnus X-1 and its companion HDE 226868.

Distant background stars show up as isolated bright (yellow or white) pixels. The difference in intensity seen in these pixels is due to fluctuations in the background.

Look closely at the image to see if you can find the volume occupied by Cygnus X-1. Here's a close-up that might help. Use the figure 3!

Question 8. What is the Cygnus X-1 volume in cubic meters from the plot in3? Explain your answer.

2.4 3rd Method: Kepler laws

You learned in class that it's generally not possible to determine the mass of a single star from its light alone. However, Cygnus X-1 is in a binary star system with a companion star, HDE 226868, a BO supergiant star. Being part of binary means that Cyg X-1 has a companion that it orbits. You recall that Kepler's Laws can be applied to any orbiting bodies, so perhaps you could use them to find the mass of Cyg X-1.

Going back to your astronomy text book, you refresh your memory of Kepler's Laws:

- Kepler's First Law, the law of orbits, states that all orbiting bodies move in elliptical orbits with the center of mass at one focus.
- Kepler's Second Law, the law of equal areas, states that the line between the stars (called the radius vector) sweeps out equal ares in equal periods of time.
- Kepler's Third Law, the law of periods, states that the square of the star's orbital period is proportional to the cube of its mean distance from the center of mass cubed.

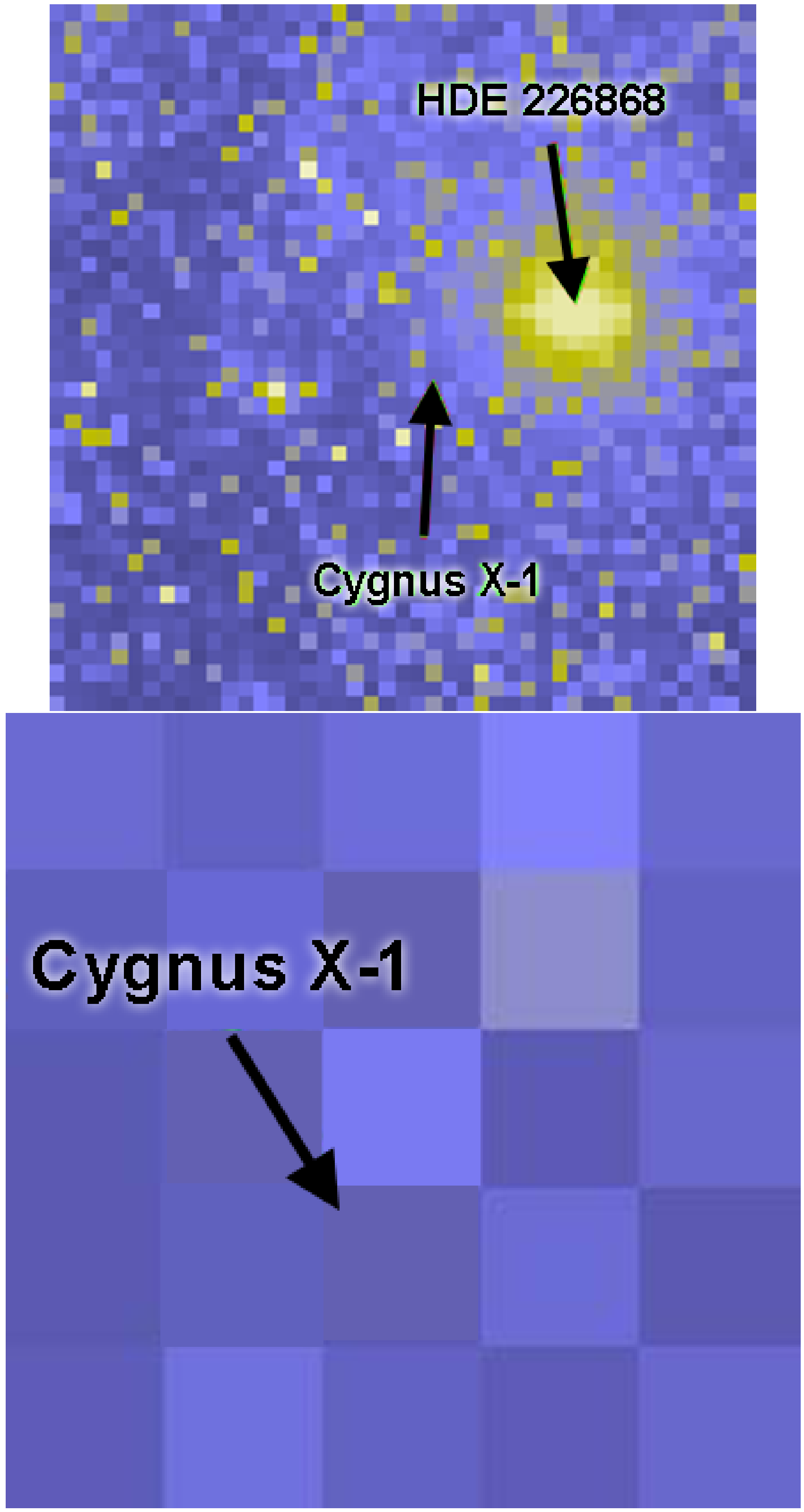

Figure 3: Pixel volume for HD226868 and Cyg X-1.

Exploring your astronomy book a little further, you find the following equation tucked away in one of the "further reading" boxes:

$$f(M) = \frac{M^3 \sin^3 i}{(m+M)^2} = \frac{Tv^3}{2\pi G_N}$$

where

- M is the mass of the object of interest.
- m is the mass of the companion object.
- T is the orbital period.
- v is the orbital velocity of the object of mass m.
- i is the inclination of the orbit with respect to the observer.
- $G = G_N$ is the gravitational constant.

The equation seems promising, if only you could figure out a few of those variables for Cygnus X-1 and its companion. The deduction of the above generalized Kepler third law is as follows:

Your astronomy book goes through a detailed derivation of the equation to find the mass of a star in a binary system. But first, it says, you need to derive Kepler's Third Law.

Consider two bodies in circular orbits about each other, with masses $M = M_1$ and $m = M_2$ and separated by a distance, a. The diagram below, shows the two bodies at their maximum separation. The distance between the center of mass and m_1 is a_1 and between the center of mass and m_2 is a_2.

The center of mass equation implies

$$M_1 a_1 = M_2 a_2$$

$$a_1 = \frac{M_2}{M_1} a_2$$

$$a = a_1 + a_2$$

So,

$$a = \frac{M_2}{M_1} a_2 + a_2 = \left(\frac{M_1 + M_2}{M_1} \right) a_2$$

$$a_2 = \frac{M_1}{M_1 + M_2} a$$

For the gravitational law and the centripetal force equation, we get

$$F = G\frac{M_1 M_2}{a^2} = M_2 a_2 \omega^2$$

and then

$$G\frac{M_1 M_2}{a^2} = M_2 \frac{M_1}{M_1 + M_2} a\omega^2$$

or

$$\boxed{G(M_1 + M_2)T^2 = a^3}$$

Moreover, you can also obtain from this what we want to prove, using that for every circular orbit $Tv_1 = 2\pi a_1$, $Tv_2 = 2\pi a_2$, and therefore

$$a = \frac{T}{2\pi}(v_1 + v_2)$$

or equivalently

$$\frac{T^2}{\left[\frac{T}{2\pi}(v_1 + v_2)\right]^3} = \frac{4\pi^2}{G(M_1 + M_2)}$$

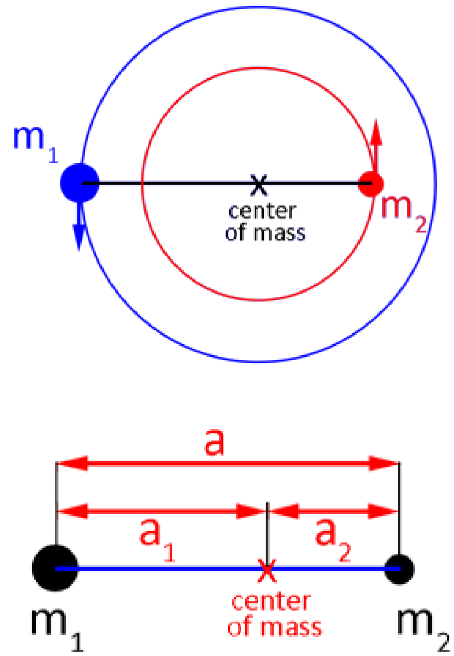

Figure 4: Binary orbits and the center of mass.

and finally

$$M = M_1 + M_2 = \frac{T(v_1 + v_2)^3}{2\pi G}$$

If we could see both stars in the binary system, this equation would work fine. However, there are times when we can only see one – for example, if there is a star in orbit with a black hole. For a circular orbit, we have:

$$M_1 v_1 = M_2 v_2$$

$$v_1 = \frac{M_2}{M_1} v_2$$

$$v_2 = \frac{M_1}{M_2} v_1$$

So,

$$M_1 + M_2 = \frac{T\left(v_1 + \frac{M_1}{M_2} v_2\right)^3}{2\pi G}$$

$$f(M_1, M_2) = \frac{M_2^3}{(M_1 + M_2)^2} = \frac{T v_1^3}{2\pi G}$$

Q.E.D. Or almost! $v_1 = v_{obs}/\sin i = K/\sin i$, where $v_{obs} = K = v$ is usually referred as the orbital semiamplitude for historical reasons (really it is an amplitude!). Therefore, we get

$$\boxed{F(M_1, M_2) = \frac{M_2^3 \sin^3 i}{(M_1 + M_2)^2} = \frac{T K^3}{2\pi G}}$$

This formula is very imporant in astrophysics for exoplanets, binary stars and generally speaking binary systems (BBH, BH-NS, NS-NS,...). It can be generalized to include the orbital eccentricity e and not to restrict ourselves to circular orbits, as follows

$$\boxed{F(M_1, M_2) = \frac{M_2^3 \sin^3 i}{(M_1 + M_2)^2} = \frac{T K^3}{2\pi G} \left(1 - e^2\right)^{3/2}}$$

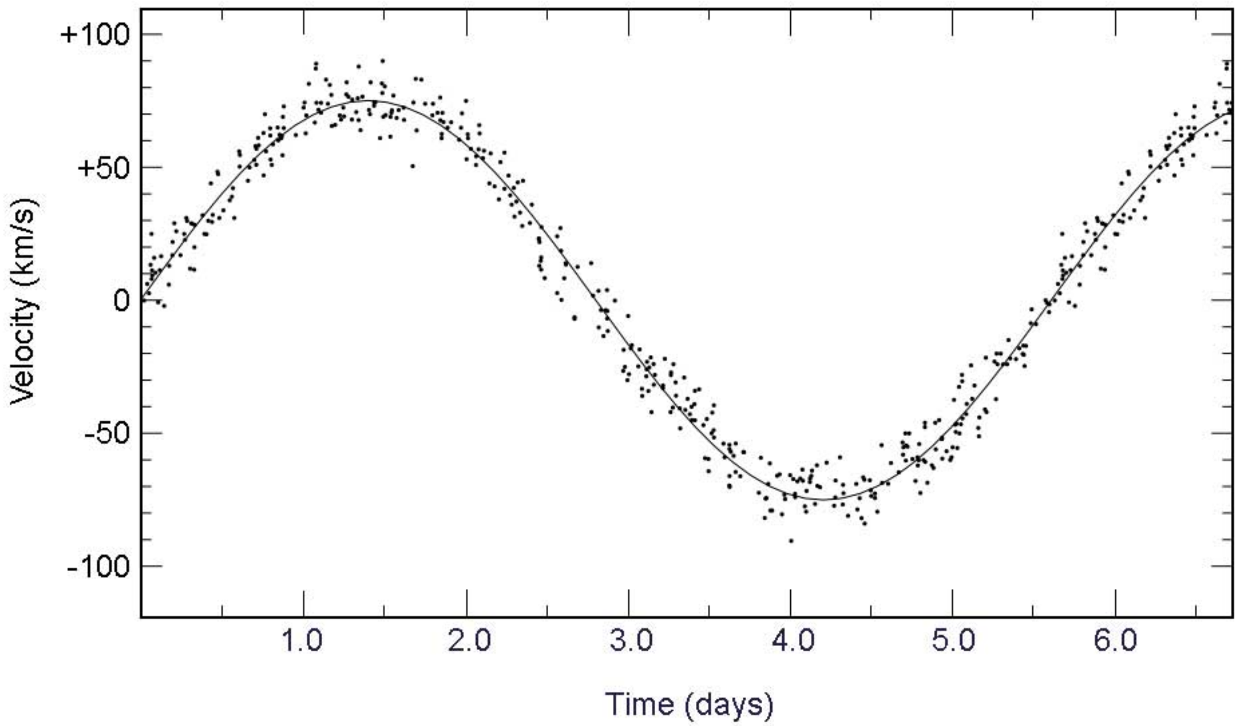

Figure 5: Cygnus X-1 radial velocity curve. Velocity curve for Cygnus X-1's optical companion. (Credit: Brocksopp, Tarasov, Lyuty, and Roche, Astronomy & Astrophysics, 343: 861 (1999)).

Astronomers can learn a number of things about objects in space by looking at subtle details and changes in the light we receive from those objects. The plot below shows the velocity of Cygnus X-1's optical companion as a function of time. The plot was generated using data from the optical spectrum of the star. The radial velocity we observe changes over time as the star orbits around Cygnus X-1. When the star is moving away from us, the velocity is positive. When the star is moving toward us, the velocity is negative.

From this plot it is possible to determine the tangential velocity and the period of the optical companion. Click on the image for a larger view. It might help to print out the plot so that you can draw on it to find the information you need.

Don't forget to make a note of the values for the period and velocity that you find – you'll need it to solve the mass equation!

Evidence for both the binary nature of HDE226868 and the link with Cygnus X-1 came when the Copernicus satellite took a closer look at the X-ray source. It was discovered that the overall intensity of the X-rays drops off slightly every 5.6 days – the same time it take for HDE226868 to make one revolution around its partner. Now, the X-rays do not drop off completely, so the X-ray source doesn't move completely behind the star from our point-of-view. At the same time, the X-rays do diminish, so the star does partially block the X-ray source as they orbit. This gives us a clue as to its inclination.

As a refresher, if a system is face-on, or the inclination is 0^{o} the X-ray source would never be blocked.

Since the X-ray source is partially blocked on each orbit, the system must be inclined at some angle between 0° and 90°. The best estimate for the inclination angle of the Cyg X-1 system is $\sim 30°$, though we may have to wait for future missions to get a more definitive answer.

If you read the background on Cygnus X-1, then you learned that the black hole's companion is a blue supergiant star classified as a type OB star. Your textbook talks at length about stellar classifications. Type O stars are the hottest while Type B are the second hottest stars out there. Type OB stars are defined as Type O or B stars that are found in groupings that all formed from the same cloud of gas.

Flipping back through your book to the section on stellar classifications, it says that Type O stars have masses between 15 and 90 times the mass of our sun and Type B stars have masses between 2 and 16 times the mass of our sun. That's quite a range of masses!

As a type OB star, it is likely on the lower end of the Type O mass rather than the higher end. You decide you can narrow down the mass to around 30 times the mass of our sun, or 30 solar masses (usually shortened to $30M_\odot$, since $\odot$ is the symbol for the sun).

Don't forget to make a note of this – you'll need it to solve the equation!

Once you've measured or estimated all of the observables, you just need to solve the mass equation. It would help to rearrange the equation above to end up with a cubic equation of the form $x^3 + ax^2 + bx + c = 0$, which can then be solved.

2.5 Solving a cubic equation

You might remember solving quadratic equations in middle school using the quadratic formula. Cubic equations can be solved with a similar formula, though the formula is a bit more complicated.

Cubic equations are polynomial equations of the form:

$$Ax^3 + Bx^2 + Cx + D = 0$$

or equivalently, if $A \neq 0$,

$$x^3 + ax^2 + bx + c = 0$$

To find out a real solution, you can proceed as follows:

- First compute the following two quantities from the coefficients a, b, and c:

$$Q = \frac{3b - a^2}{9}$$

$$R = \frac{9ab - 27c - 2a^3}{54}$$

- Secondly, from these values of Q, R, calculate

$$S = \left(R + \sqrt{Q^3 + R^2}\right)^{1/3}$$

$$T = \left(R - \sqrt{Q^3 + R^2}\right)^{1/3}$$

- Compute the real solution with

$$\boxed{x_1 = S + T - \frac{a}{3}}$$

Start with the binary mass equation:

$$M^3 \sin^3 i/(m + M)^2 = Pv^3/(2\pi G)$$

Then rearrange to get:

$$M^3 \sin^3 i = \frac{Pv^3}{(2\pi G)}(m + M)^2$$

Then expand the right-hand side and rearrange to get this:

$$M^3 - \frac{Pv^3}{(2\pi G \sin^3 i)} M^2 - 2m \frac{2Pv^3}{(2\pi G \sin^3 i)} M - \frac{Pv^3}{(2\pi G \sin^3 i)} m^2 = 0$$

Now the equation is in the form of a cubic equation with:

$$a = -Pv^3/(2\pi G \sin^3 i)$$
$$b = -2Pv^3/(2\pi G \sin^3 i)$$
$$c = -Pv^3 m^2/(2\pi G \sin^3 i)$$

Notice that each of those has the following quantity:

$$Pv^3/(2\pi G \sin^3 i)$$

In fact, it will make things easier if we just define that:

$$D = Pv^3/(2\pi G \sin^3 i)$$

Then,

$$a = -D$$
$$b = -2mD$$
$$c = -Dm^2$$

Using the following quantities, and converting to meters-kilograms-seconds units:

$$i = 30$$
$$P = 5.6 days$$
$$v = K = 75 km/s = 7.5 \cdot 10^4 m/s$$
$$m = 30 M_\odot = 30 \cdot (1.989 \cdot 10^{30}) kg = 5.7 \cdot 10^{31} kg$$

Your values might be a little different from those above – that's okay, as long as they aren't too far off. Substituting values in, then¡

$$D = 4.8 \cdot 10^5 s (7.5 \cdot 10^4 m/s)^3 / (2 \cdot \pi 6.67 \cdot 10^{-11} m^3/kg/s^2 \cdot \sin^3 30) = 3.9 \cdot 10^{30} kg$$

Numerically the values for the coefficients are:

$$a = -3.9 \cdot 10^{30} kg$$
$$b = -4.4 \cdot 10^{62} kg^2$$
$$c = -1.3 \cdot 10^{94} kg^3$$

Check that dimensionally, a, b, c are correct, and that the numbers are reasonable.

You are now ready to compute the quantities necessary to find the real cubic root.

One additional hint: it will make things a lot simpler if you keep things in terms of solar masses. If you do, then,

$$D = 2.0 M_\odot$$

And,

$$a = -2.0 M_\odot, \quad b = -120 M_\odot^2, \quad c = -1800 M_\odot^3$$

Then, just carry through those solar mass symbols through all of your calculations. The final answer is supposed to be in terms of solar mass anyway, and this way you don't have to deal with the huge exponents.

3 Conclusion

Quote the final value of the Cygnus X-1 invisible companion of HD 226868 in kilograms, grams and solar masses. Check your solution with the real values and comment about the accuracy and precision of your final mass result. What are the more important sources of error of your value? Search for current investigations about this binary system.

https://imagine.gsfc.nasa.gov/features/yba/CygX1_mass/

EXTRA POINT: With figure 6 data, build up your own radial velocity curve and try to fit it to a sinusoidal function. Use the software/spreadsheet you wish, but explain step to step what you do.

References

[1] https://imagine.gsfc.nasa.gov/features/yba/CygX1_mass/

JDh (2400000+)	V_r (km s^{-1})	r.m.s. error
50615.4956	−48.5	1.0
50623.4675	66.4	1.4
50624.4585	−14.3	1.2
50625.4553	−63.3	2.3
50626.4792	−72.7	1.3
50648.4623	−78.5	1.3
50649.5196	−7.5	1.8
50650.5054	56.8	1.3
50651.5033	62.9	1.4
50653.5095	−65.4	1.1
50658.4643	−26.1	1.0
50660.3914	−52.1	1.6
50661.3882	33.5	1.7
50663.4452	4.2	1.0
50667.4747	77.3	1.6
50668.4164	66.2	1.7
50669.4382	10.0	2.5
50675.3550	−44.7	1.2
50676.4727	−86.0	1.4
50677.4751	−28.8	1.4

Figure 6: Radial velocity HD 226868-CygX1 system.